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FUNDAMENTAL SOLUTION OF THE STEADY OSCILLATIONS EQUATIONS IN COUPLE STRESS MICROPOLAR VISCOELASTIC THERMOELASTIC SOLID

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In this article, we have studied the case of steady oscillations to solve the system of differential equations by using elementary functions in the theory of microploar couple stress viscoelastic solid. Four waves were discovered: two coupled longitudinal waves and two coupled transverse waves that propagate at various wave speeds. The penetration depth, specific loss, attenuation coefficients, and phase velocity–all of which have been computed numerically and plotted graphically for the LS and GL models–are affected by couple stresses and viscous parameters. Specific cases are created and contrasted with the earlier outcomes.

Key words: fundamental solution, plane wave, steady oscillations, couple stress, viscoelastic, micropolar, thermoelastic.

1. Introduction

The most relevant features of linear elasticity theory are applied in the steel stress analysis. The idea of couple stress was postulated as a result of the classical theory's failure to explain vibration in granular and multimolecular bodies. Voigt [1] was the first to introduce this concept. Cosserat and Cosserat [2] proposed theory of non linear asymmetric elasticity with couple stresses. In the original Cosserat theory, the kinematical quantities were the displacements and a material microrotation, hypothesized to be independent of the continuum mechanical rotation. Eringen and Suhubi [3] and Suhubi and Eringen [4] postulated micro-elasticity theory , in which the microelements have intrinsic motions. Eringen [5] investigated the hypothesis of micropolar elasticity by examining microstructures.

Eringen [6-7] and Nowacki [8] proposed the micropolar theory. Ciarletta [9] proposed the micropolar theory of thermoelasticity without energy dissipation. Chandrasekhariah [10] investigated thermoelastic theory by using dual-phase-lag effects. Also a huge contribution to this theory was given by Biot [11]. An application of flexible micropolar material that combines a dual-phase lag thermoelastic model (DPL) with a consistent focus on the idea of a memory-dependent derivative was investigated by Abouelregal *et al.* [12]. Also, Abouelregal and Rashid [13] investigated many problems related to deformation in a micropolar material. Many researchers have done impressive work on wave propagation in isotropic and anisotropic media with additional parameters like micropolarity, the thermal field with fraction-order derivative, initial stress, voids and diffusion. Yadav [14-17] investigated numerous issues related to micopolar theory under different conditions. Singh *et al.* [18] and Singh and Yadav [19] investigated the propagation of plane waves in the theory of micropolar thermoelasticity with diffusion.

Lord-Shulman determined the theory of generalised thermoelasticity which explained how Fourier law took place of Maxwell-Catteneo law with one relaxation time. Green-Lindsay theory also recognized by the thermoelasticity theory depends upon temperature rate or ther moelasticity theory with two relaxation times. Three generalisations of this theory were examined by Hetnarski and Ignaczak [20]: (i) coupled theory (CT),

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(ii) Lord-Shulman theory (LST) [21] and (iii) Green-Lindsay theory (GLT) [22]. Abouelregal [23-24] used modified couple stress model to analyze the effect of size-dependent on thermal interactions in rotating nanobeams whose properties change with temperature and also shows the effect of non-local modified couple stress theory on the responses of axially moving thermoelastic nano-beams.

In order to determine the basic solution for the thermoelastic theory using the boundary integral method, we must solve the differential equations. Lord-Shulman determined the theory of generalised thermoelasticity which explained how Fourier law took place in Maxwell-Catteneo law with one relaxation time. Green-Lindsay theory also recognized by the thermoelasticity theory depends upon temperature rate or their elasticity theory with two relaxation times. Yadav [25-26] and Yadav *et al.* [27] for their efforts and their overview of utilizing thermoelascity in scientific research and their applications as well. The basic solutions of the coupled thermoelastic issue for small timeframes were presented by Hetnarski [28]. Iesan [29] investigated a solution on the theory of thermoelasticity without energy dissipation. The fundamental solutions in the microcontinuum field theories were proposed by Svanadze [30-31]. Hormander [32-33] and Kumar et al. [34] examined the useful data by resolving the basic equations of the different kinds of equations. Sahrawat *et al.* [35] and Poonam [36] investigated numerous issues pertaining to wave transmission. However, from best of author knowledge, no study has been performed for predicting the combining effect of viscous and couple stress parameters on fundamental solution and plane wave propagation under thermoelastic theories. The propagation of waves at an interface is widely used in many fields including geophysics and engineering.

This work is primarily concerned with the propagation of plane waves and the fundamental solution of the system of equations in couple stress micropolar viscoelastic generalised thermoelastic solid medium. There are two coupled longitudinal and transverse waves moving at different speeds. For both the LS- and GL-models, the specific loss, penetration depth, attenuation coefficients, and phase velocity have been computed both graphically and quantitatively in relation to wave number .These parameters are greatly influenced by the couple stress and viscous parameters. Additionally, some fundamental solution properties are constituted and some particular cases are deduced from the previous investigation.

2. General equation

Let $x=(x_1,x_2,x_3)$ be in 3-D Euclidean space E^3, t is the time variable and $D_x \equiv \left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$.

Following Baksi *et al.* [37], Eringen [38] and Eringen [39], the general equations for micropolar couple stress viscoelastic thermoelastic solid are given as

$$t_{ij} = -\tau_m \delta_{ij}\theta + \lambda_3 \delta_{ij}\epsilon_{rr} + (\mu_3 + K_3)\epsilon_{kl} + \mu_3 \epsilon_{lk}\,, \tag{2.1}$$

$$m_{ij} = \alpha_3 \gamma_{rr}\delta_{ij} + \beta_3 \gamma_{ij} + \gamma_3 \gamma_{ji}\,, \tag{2.2}$$

$$\eta = \vartheta \epsilon_r r \delta_{ij} + a\theta\,, \tag{2.3}$$

$$q_i = k_1\left(1 - t_0 \delta_{ij}\frac{\partial}{\partial t}\right)\theta_{,i}\,, \tag{2.4}$$

$$\tau_m = \left(\vartheta_1 + \vartheta_2 t_1 \delta_{k2}\frac{\partial}{\partial t}\right), \tag{2.5}$$

$$\lambda_3 = \lambda_2 + \lambda_2^*\frac{\partial}{\partial t}, \quad \mu_3 = \mu_2 + \mu_2^*\frac{\partial}{\partial t}, \quad K_3 = K_2 + K_2^*\frac{\partial}{\partial t}\,,$$

$$\alpha_3 = \alpha_2 + \alpha_2^* \frac{\partial}{\partial t}, \quad \beta_3 = \beta_2 + \beta_2^* \frac{\partial}{\partial t}, \quad \gamma_3 = \gamma_2 + \gamma_2^* \frac{\partial}{\partial t}$$

and $\epsilon_{ij} = e_{ij} + e_{ijk}\left(R_k - \phi_k\right)$ and $\gamma_{ij} = \phi_{i,j}$ denote the micropolar tensor and curvature tensor respectively; R and ϕ denote macrorotation vector and microrotation vector respectively. The cases $k = 1,2$ correspond to LS theory and the GL theory respectively. When both t_0 and t_1 vanish, it represents to the coupled theory (CT) of thermoelasticity.
If $R = 0$, the micropolar tensor and curvature tensor become

$$\epsilon_{ij} = e_{ij} + e_{ijk}\phi_k, \quad \gamma_{ij} = \phi_{i,j} . \tag{2.6}$$

Now, we introduce the following dimensionless quantities

$$x_i^{'} = \frac{x_i}{l_0}, \quad u_i^{'} = \frac{u_i}{l_0}, \quad \phi_i^{'} = \frac{\phi_i}{l_0}, \quad t' = \frac{c_0 t}{l_0}, \quad T' = \frac{\theta}{T_0}, \quad t'_{ij} = \frac{t_{ij}}{l_0 \vartheta T_0}, \quad m'_{ij} = \frac{m_{ij}}{l_0 \vartheta T_0} \tag{2.7}$$

where c_0, l_0 and T_0 are standard velocity, standard displacement and standard temperature respectively.
The basic equation of the elastic medium are given by

$$t_{ji,j} + \rho\left(f_l - \ddot{u}_i\right) = 0 , \tag{2.8}$$

$$m_{ji,j} + \epsilon_{imn}t_{mn} + \rho\left(l_i - j_0\ddot{\phi}_i\right) = 0 , \tag{2.9}$$

$$-\rho T_0\dot{\eta} + q_{i,i} + \rho h = 0 . \tag{2.10}$$

Using Eqs (2.1)-(2.7) in Eqs (2.8)-(2.10), general equations for micropolar couple stress viscoelastic generalized thermoelastic solid without heat source, body forces and body couples are obtained as

$$\begin{gathered}\left(\mu_2 + K_2\right)\nabla^2 \boldsymbol{u} + \left(\mu_2^* + K_2^*\right)\nabla^2 \dot{\boldsymbol{u}} + \left(\lambda_2 + \mu_2\right)\nabla\left(\nabla\cdot\boldsymbol{u}\right) + \left(\lambda_2^* + \mu_2^*\right)\nabla\left(\nabla\cdot\dot{\boldsymbol{u}}\right) + \\ + K_2\nabla\times\phi + K_2^*\nabla\times\dot{\phi} - \vartheta_1\nabla T - \vartheta_2\nabla\dot{T} = \ddot{\boldsymbol{u}}\end{gathered} \tag{2.11}$$

$$\begin{gathered}\left(\alpha_2 + \beta_2\right)\nabla\left(\nabla\cdot\phi\right) + \left(\alpha_2^* + \beta_2^*\right)\nabla\left(\nabla\cdot\dot{\phi}\right) + \gamma_2\nabla^2\phi + \gamma_2^*\nabla^2\dot{\phi} + \\ + K_2\nabla\times\boldsymbol{u} + K_2^*\nabla\times\dot{\boldsymbol{u}} - 2K_2\phi - 2K_2^*\dot{\phi} = r_1\ddot{\phi}\end{gathered} \tag{2.12}$$

$$\left(1 + n_1\delta_{lk}\frac{\partial}{\partial t}\right)\dot{T} + \left(\frac{\vartheta_1}{\tau_3} + \frac{\vartheta_1}{\tau_2}\delta_{lk}\right)\dot{\boldsymbol{u}}_{\boldsymbol{i},i} = k_1\nabla^2 T . \tag{2.13}$$

For 2-D problem, taking

$$\boldsymbol{u} = \left(u_1, 0, u_3\right).$$

Utilising Helmholtz decomposition theorem, ***u*** can be expressed as

$$\boldsymbol{u}=\nabla\sigma+\nabla\times\boldsymbol{\Psi},\quad \nabla\cdot\boldsymbol{\Psi}=0 \tag{2.14}$$

where $\sigma(x_1,x_3,t)$ and $\boldsymbol{\Psi}(x_1,x_3,t)$ are the potential function.
Using Eq.(2.14), Eqs (2.11)-(2.13) take the form

$$-c_1^2T+(\lambda_3+2\mu_3+K_3)\nabla^2\sigma-\rho\ddot{\sigma}=0, \tag{2.15}$$

$$(\mu_3+K_3)\nabla^2\boldsymbol{\Psi}+K_3\nabla\phi-\rho\ddot{\boldsymbol{\Psi}}=0, \tag{2.16}$$

$$(\alpha_3+\beta_3+\gamma_3)\nabla^2\phi+K_3\nabla\times\nabla\times\boldsymbol{\Psi}-2K_3\phi-r_1\ddot{\phi}=0, \tag{2.17}$$

$$k_1\nabla^2T-c_3^2\nabla^2\dot{\sigma}-c_4^2\dot{T}=0 \tag{2.18}$$

where, $c_1^2=\left(\vartheta_1+\vartheta_2\delta_{2k}\dfrac{\partial}{\partial t}\right)$, $\quad c_3^2=\left(\dfrac{\vartheta_1}{\tau_3}+\dfrac{\vartheta_1\delta_{1k}}{\tau_2}\dfrac{\partial}{\partial t}\right)$, $\quad c_4^2=\left(1+n_1\delta_{1k}\dfrac{\partial}{\partial t}\right)$.

3. Plane wave

The solution of the system can be taken as

$$\{\sigma,\ \phi,\ \boldsymbol{\Psi},\ T\}=(\sigma',\ \phi',\ \boldsymbol{\Psi}',\ \theta)\exp\{ik(\boldsymbol{n}\cdot\boldsymbol{r}-ct)\} \tag{3.1}$$

where k signifies wave number, $\sigma', \phi', \boldsymbol{\Psi}', \theta$ are time dependent stationary values and coordinates $\boldsymbol{r}=x_s\ (s=1,\ 3)$ and c denotes phase velocity.
Using Eq.(3.1) in Eqs (2.15)-(2.18), we obtain

$$c'^2_1\,\theta+\left[b_1k^2+\rho c^2k^2\right]\sigma'=0, \tag{3.2}$$

$$\left((\mu'_3+K'_3)+\rho c^2\right)k^2\boldsymbol{\Psi}'+K'_3\,\phi'=0, \tag{3.3}$$

$$-\left(b_2-r_1c^2\right)k^2\phi'+K'_3\,k^2\boldsymbol{\Psi}'-2K'_3\,\phi'=0, \tag{3.4}$$

$$\left(-k_1k^2+c'^2_4\,\imath kc\right)\theta+\imath\omega k^2c'^2_3\,\sigma'=0 \tag{3.5}$$

where $\quad \lambda'_3=\lambda_2-i\omega\lambda_2^*,\quad \mu'_3=\mu_2-i\omega\mu_2^*,\quad K'_3=K_2-i\omega K_2^*,\quad \alpha'_3=\alpha_2-i\omega\alpha_2^*,$

$\beta'_3=\beta_2-i\omega\beta_2^*,\quad \gamma'_3=\gamma_2-i\omega\gamma_2^*,\quad b_1=\lambda'_3+2\mu'_3+K'_3,\quad b_2=\alpha'_3+\beta'_3+\gamma'_3,$

$c'^2_1=(\vartheta_1-\imath\omega\vartheta_2\delta_{2k}),\quad c'^2_3=\left(\dfrac{\vartheta_1}{\tau_3}-i\omega\dfrac{\vartheta_1\delta_{1k}}{\tau_2}\right),\quad c'^2_4=(1-i\omega n_1\delta_{1k}).$

On solving Eqs (3.2) and (3.5), yield the polynomial equation in k as

$$B_1 k^4 + B_2 k^2 + B_3 = 0 \tag{3.6}$$

where $$B_1 = k_1 b_1, \quad B_2 = -k_1 \rho\omega^2 - i\omega b_1 c_4'^2, \quad B_3 = -i\omega\left(c'^2_3\, c'^2_1 + c'^2_4\, \rho\omega^2\right).$$

Solving Eq.(3.6), we obtain four complex roots, $\pm k_i$, $(i = 1,2)$. So, there exist a set of coupled longitudinal waves. Utilising Eq.(3.1) in (2.14), we are able to determine the nature of these waves as $\boldsymbol{u}$

$$\boldsymbol{u} = \iota k \sigma' \boldsymbol{n} \exp\left\{\iota k\left(\boldsymbol{n}.\boldsymbol{r} - ct\right)\right\}.$$

This indicates that $\boldsymbol{u}$ is parallel to the $\boldsymbol{n}$ vector. Therefore, we identify a pair of coupled longitudinal waves: the longitudinal thermal wave and the longitudinal displacement wave.
By solving Eqs (3.3) and (3.4), we obtain a polynomial equations in k as

$$B_4 k^4 + B_5 k^2 + B_6 = 0 \tag{3.7}$$

where

$$B_4 = \left(\mu'_3 + K'_3\right) b_2, \quad B_5 = \left(\mu'_3 + K'_3\right)\left(2K'_3 - r_1\omega^2\right) + b_2 \rho\omega^2 - K'^2_3, \quad B_6 = \rho\omega^2\left(2K'_3 - r_1\omega^2\right).$$

From Eq.(3.7), we find four complex roots, $\pm k_i$, $(i = 3,4)$ and according to these roots, we find a set of coupled transverse waves.
Using Eq.(3.1) in Eq.(2.16), we determine the behaviour of these waves by observing that the potential $\boldsymbol{\Psi}$ is transverse in nature since it is normal to the direction of wave propagation, $\boldsymbol{n}$.
Transverse displacement wave and transverse microrotational wave are two set of coupled transverse waves.

(1) **Penetration depth:** The penetration depth (P_s) are provided by

$$P_s = \frac{1}{\left|Im\left(k_s\right)\right|}.$$

(2) **Specific loss:** The specific loss R_s are represented as

$$R_s = \left(\frac{W'_s}{W_s}\right) = 4\pi\left|\frac{Re\left(k_s\right)}{Im\left(k_s\right)}\right|$$

where W'_s = energy dissipated and W_s = elastic energy coefficient.

(3) **Attenuation coefficients:** The attenuation coefficients (Q_s) are provided by

$$Q_s = Im\left(k_s\right).$$

(4) **Phase velocity:** The phase velocities $\left(V_s\right)$ are represented as

$$V_s = \frac{\omega}{\left|Re\left(k_s\right)\right|} s = 1,2,3,4 .$$

4. Steady oscillations

Assuming a harmonic time variation for $\boldsymbol{u}$**,** T and the microrotation vector $\boldsymbol{\upsilon}$,

$$\left\{\boldsymbol{u}(x,t),\phi(x,t),T(x,t)\right\} = \mathrm{Re}\left\{\left(\boldsymbol{u}_1(x),\phi_1(x),\theta(x)\right)\exp(-\iota\omega t)\right\} .$$

From the Eqs (2.11)-(2.13), we find these equations of steady oscillations

$$\left(\lambda'_3+\mu'_3\right)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}_1 + \left(\mu'_3+K'_3\right)\nabla^2\boldsymbol{u}_1 + K'_3\, curl\,\phi_1 - c'^2_1\,\mathrm{grad}\,\theta + \rho\omega^2\boldsymbol{u}_1 = 0 , \tag{4.1}$$

$$\left(\alpha'_3+\beta'_3\right)\mathrm{grad}\,\mathrm{div}\,\phi_1 + \gamma'_3\,\nabla^2\phi_1 + K'_3\, curl\,\boldsymbol{u}_1 + \mu'_1\,\phi_1 = 0 , \tag{4.2}$$

$$a_0\theta + m_0\,\mathrm{div}\ \boldsymbol{u}_1 + k_1\nabla^2\theta = 0 \tag{4.3}$$

where $\quad \mu'_1 = r_1\omega^2 - 2K'_3\; a_0 = i\omega c'^2_4\; m_0 = c'^2_3\,\iota\omega .$

Let us define a matrix differential operator

$$M\left(\boldsymbol{D_x}\right) = \left\| M_{qn}\left(\boldsymbol{D_x}\right)_{7\times 7}\right\|$$

where $\quad M_{lj}\left(\boldsymbol{D_x}\right) = \left[\left(\mu'_3+K'_3\right)\nabla^2 + \rho\omega^2\right]\delta_{lj} + \left(\lambda_{3'}+\mu'_3\right)\dfrac{\partial^2}{\partial x_l \partial x_j} ,$

$$M_{l+3,j}\left(\boldsymbol{D_x}\right) = M_{l,j+3}\left(D_x\right) = K'_3\sum_{r=1}^{3}\epsilon_{lrj}\frac{\partial}{\partial x_r} ,$$

$$M_{l7}\left(\boldsymbol{D_x}\right) = -c'^2_1\frac{\partial}{\partial x_l} M_{l+3,7}\left(\boldsymbol{D_x}\right) = M_{7,j+3}\left(\boldsymbol{D_x}\right) = 0 ,$$

$$M_{l+3,j+3}\left(\boldsymbol{D_x}\right) = \left[\gamma'_3\,\nabla^2 + \mu'_1\right]\delta_{lj} + \left(\alpha'_3+\beta'_3\right)\frac{\partial^2}{\partial x_l \partial x_j} ,$$

$$M_{7l}\left(\boldsymbol{D_x}\right) = m_0\frac{\partial}{\partial x_l} M_{77}\left(\boldsymbol{D_x}\right) = k_1\nabla^2 + a_0\, l, \quad j = 1,2,3 .$$

We can write Eqs (4.1)-(4.3) as

$$M\left(\boldsymbol{D_x}\right)\boldsymbol{U_x}=0$$

where $\boldsymbol{U}$ is the vector function with seven components expressed in terms of $\left(\boldsymbol{u_1},\ \phi_1,\theta\right)$.
Let us consider elastic constants satisfy the condition.

$$\gamma'_3\,k_1\left(\mu'_3+K'_3\right)\left(\lambda'_3+2\mu'_3+K'_3\right)\left(\alpha'_3+\beta'_3+\gamma'_3\right)\neq 0\,, \tag{4.4}$$

$\boldsymbol{M}$ is an elliptic differential operator.

Definition: The matrix $G\left(x\right)=\left\|G_{lj}\right\|_{7\times 7}$ satisfying Eqs (4.1)-(4.3) (basic matrix of operator $\boldsymbol{M}$) is the fundamental solution.

$$\boldsymbol{M}\left(D_x\right)G\left(x\right)=\delta\left(x\right)\boldsymbol{I} \tag{4.5}$$

where $\qquad x\in E^3\,,\qquad I=\left\|\delta_{lj}\right\|_{7\times 7}\,.$

5. Fundamental solution

For the fundamental solution ,we take

$$\left(\lambda'_3+\mu'_3\right)\operatorname{grad}\operatorname{div}\boldsymbol{u_1}+\left(\mu'_3+K'_3\right)\nabla^2\boldsymbol{u_1}+K'_3\,curl\,\phi_1+m_0\operatorname{grad}\theta+\rho\omega^2\boldsymbol{u_1}=\boldsymbol{F'}\,, \tag{5.1}$$

$$\left(\alpha'_3+\beta'_3\right)\operatorname{grad}\operatorname{div}\phi_1+\gamma'_3\nabla^2\phi_1+K'_3\,curl\,\boldsymbol{u_1}+\mu'_1\,\phi_1=\boldsymbol{F''}\,, \tag{5.2}$$

$$\left(k_1\nabla^2+a_0\right)\theta-c'^2_1\operatorname{div}\,\boldsymbol{u_1}=F_0 \tag{5.3}$$

where scalar functions are indicated by F_0 and vector functions by $\boldsymbol{F'}$ and $\boldsymbol{F''}$.
Equations (5.1)-(5.3) can be represented as

$$\boldsymbol{M^T}\left(\boldsymbol{D_x}\right)\boldsymbol{U}\left(x\right)=\boldsymbol{F}\left(x\right). \tag{5.4}$$

As F is combination of seven terms ($\boldsymbol{F'}$, $\boldsymbol{F''}$, F_0), and $\boldsymbol{T}$ is transpose.
Applying "div" to Eqs (5.1) and (5.2), yield

$$\left[\mu_0\nabla^2+\rho\omega^2\right]\operatorname{div}\boldsymbol{u_1}+m_0\nabla^2\theta=\operatorname{div}\boldsymbol{F'}\,,$$

$$-c'^2_1\operatorname{div}\boldsymbol{u_1}+\left(k_1\nabla^2+a_0\right)\theta=F_0\,, \tag{5.5}$$

$$\left[\gamma_0\nabla^2+\mu'_1\right]\operatorname{div}\phi_1=\operatorname{div}\boldsymbol{F''} \tag{5.6}$$

where $\qquad \mu_0=\lambda'_3+2\mu'_3+K'_3$ and $\gamma_0=\alpha'_3+\gamma'_3+\beta'_3\,.$

We can write Eqs (5.3),(5.5) and (5.6) as

$$\Lambda_1\left(\nabla^2\right)\mathrm{div}\,\boldsymbol{u}_1 = \boldsymbol{\Phi}_1, \tag{5.7}$$

$$\Lambda_1\left(\nabla^2\right)\theta = \boldsymbol{\Phi}_2 \tag{5.8}$$

where $\Lambda_1\left(\nabla^2\right) = b_1\left[\left(\mu_0\nabla^2 + \rho\omega^2\right)\left(k_1\nabla^2 + a_0\right) + m_0 c'^2_1\nabla^2\right], \quad b_1 = \dfrac{1}{(\mu_0 k_1)},$

$$\boldsymbol{\Phi}_1 = b_1\left[\left(k_1\nabla^2 + a_0\right)\mathrm{div}\,\boldsymbol{F'} - m_0\nabla^2 F_0\right], \tag{5.9}$$

$$\boldsymbol{\Phi}_2 = b_1\left[c'^2_1\,\mathrm{div}\,\boldsymbol{F'} + \left(\mu_0\nabla^2 + \rho\omega^2\right)F_0\right]. \tag{5.10}$$

It is clearly seen that

$$\Lambda_1\left(\nabla^2\right) = \left(\nabla^2 + l_1^2\right)\left(\nabla^2 + l_2^2\right) \tag{5.11}$$

where $\Lambda_1(-\chi) = 0$ (w.r.t. φ) is satisfied by l_s^2 ($s = 1,2$).
From Eq.(5.6), we have

$$\left(\nabla^2 + l_5^2\right)\mathrm{div}\,\phi_1 = \frac{1}{\gamma_0}\mathrm{div}\,\boldsymbol{F''} \tag{5.12}$$

where $l_5^2 = \mu'_1/\gamma_0$.
Multiplying Eqs (5.1) and (5.2) by the operators $\left(\gamma'_3\nabla^2 + \mu'_1\right)$ and (K'_3 curl), respectively, we obtain

$$\begin{gathered}\left(\gamma'_3\nabla^2 + \mu'_1\right)\left[(\mu'_3 + K'_3)\nabla^2\boldsymbol{u}_1 + (\lambda'_3 + \mu'_3)\,\mathrm{grad\,div}\boldsymbol{u}_1 + \rho\omega^2 u_1\right] + \\ + K'_3\left(\gamma'_3\nabla^2 + \mu'_1\right)curl\phi_1 = \left(\gamma'_3\nabla^2 + \mu'_1\right)\left(\mathrm{div}\,\boldsymbol{F'} - m_0\,grad\theta\right),\end{gathered} \tag{5.13}$$

$$K'_3\left(\gamma'_3\nabla^2 + \mu'_1\right)\mathrm{curl}\,\phi_1 = -K'^2_3\,\mathrm{curl\,curl}\,\boldsymbol{u}_1 + K'_3\,\mathrm{curl}\,\boldsymbol{F''}. \tag{5.14}$$

Since

$$\mathrm{curl\,curl}\,\boldsymbol{u}_1 = \mathrm{grad\,div}\,\boldsymbol{u}_1 - \nabla^2\boldsymbol{u}_1.$$

Therefore, from Eq.(5.13), we have

$$\begin{gathered}\left\{\left[\left(\gamma'_3\nabla^2 + \mu'_1\right)(\mu'_3 + K'_3) + K'^2_3\right]\nabla^2 + \rho\omega^2\left(\gamma'_3\nabla^2 + \mu'_1\right)\right\}\boldsymbol{u}_1 = -\left[(\lambda'_3 + \mu'_3)\right. \\ \left.\left(\gamma'_3\nabla^2 + \mu'_1\right) - K'^2_3\right]\mathrm{grad\,div}\,\boldsymbol{u}_1 + \left(\gamma'_3\nabla^2 + \mu'_1\right)\left(\mathrm{div}\,\boldsymbol{F}' - m_0\,grad\,\theta\right) - K'_3\,\mathrm{curl}\,\boldsymbol{F''}.\end{gathered} \tag{5.15}$$

Using Eqs (5.1)-(5.4) and $\Lambda_1\left(\nabla^2\right)$ applied to Eq.(5.15), we obtain

$$\begin{aligned}&\Lambda_1\left(\nabla^2\right)\left[\gamma'_3(\mu'_3+K'_3)\nabla^4+\left(\mu'_3\mu'_1+K'_3\mu'_1+K'^2_3+\rho\omega^2\gamma'_3\right)\nabla^2+\mu'_1\rho\omega^2\right]\boldsymbol{u}_1=\\&=\left(\gamma'_3\nabla^2+\mu'_1\right)\left[\Lambda_1\left(\nabla^2\right)\boldsymbol{F'}-m_0\,\mathrm{grad}\,\boldsymbol{\Phi}_2\right]-K'_3\Lambda_1\left(\nabla^2\right)\mathrm{curl}\,\boldsymbol{F}''+\\&-\left[(\lambda'_3+\mu'_3)\left(\gamma'_3\nabla^2+\mu'_1\right)-K'^2_3\right]\mathrm{grad}\,\boldsymbol{\Phi}_1\ .\end{aligned}\tag{5.16}$$

We can write Eq.(5.16) as

$$\Lambda_1\left(\nabla^2\right)\Lambda_2\left(\nabla^2\right)\boldsymbol{u}_1=\boldsymbol{\Phi'}\tag{5.17}$$

where $\Lambda_2\left(\nabla^2\right)=b_2\left[\left((\mu'_3+K'_3)\nabla^2+\rho\omega^2\right)\left(\gamma'_3\nabla^2+\mu'_1\right)+K'^2_3\nabla^2\right],\qquad b_2=\dfrac{1}{\gamma'_3(\mu'_3+K'_3)}$

and

$$\begin{aligned}&\boldsymbol{\Phi'}=b_2\left\{\left(\gamma'_3\nabla^2+\mu'_1\right)\left[\Lambda_1\left(\nabla^2\right)\boldsymbol{F'}-m_0\,\mathrm{grad}\,\boldsymbol{\Phi}_2\right]-K'_3\Lambda_1\left(\nabla^2\right)\mathrm{curl}\,\boldsymbol{F''}+\right.\\&\left.-\left[(\lambda'_3+\mu'_3)\left(\gamma'_3\nabla^2+\mu'_1\right)-K'^2_3\right]\right\}\mathrm{grad}\,\boldsymbol{\Phi}_1.\end{aligned}\tag{5.18}$$

It is evident that

$$\Lambda_2\left(\nabla^2\right)=\left(\nabla^2+l_3^2\right)\left(\nabla^2+l_4^2\right)\tag{5.19}$$

where $\Lambda_2(-\chi)=0$ (w.r.t. χ) is satisfied by l_s^2 ($s=3,4$).

From Eqs (5.1), (5.2) and (5.12), we obtain

$$\Lambda_2\left(\nabla^2\right)\left(\nabla^2+l_5^2\right)\phi_1=\boldsymbol{\Phi''}\tag{5.20}$$

where

$$\begin{aligned}&\boldsymbol{\Phi''}=b_2\left\{-K'_3\left(\nabla^2+l_5^2\right)\mathrm{curl}\,F'+\left(\nabla^2+l_5^2\right)\left[(\mu'_3+K'_3)\nabla^2+\rho\omega^2\right]\boldsymbol{F''}+\right.\\&\left.-\frac{1}{\gamma_0}\left[(\alpha'_3+\beta'_3)\left((\mu'_3+K'_3)\nabla^2+\rho\omega^2\right)-K'^2_3\right]\mathrm{grad}\,\mathrm{div}\,F''\right\}.\end{aligned}\tag{5.21}$$

From Eqs (5.17), (5.20) and (5.8), we obtain

$$\boldsymbol{\Lambda}\left(\nabla^2\right)\boldsymbol{U}(x)=\tilde{\boldsymbol{\Phi}}(\boldsymbol{x})\tag{5.22}$$

where a seven-component vector function on E^3 is represented by $\tilde{\boldsymbol{\Phi}}=\left(\boldsymbol{\Phi'},\boldsymbol{\Phi''},\boldsymbol{\Phi}_2\right)$,

$$\Lambda\left(\nabla^2\right) = \left|\left|\Lambda_{pq}\left(\nabla^2\right)\right|\right|_{7\times 7},$$

$$\Lambda_{ll}\left(\nabla^2\right) = \Lambda_1\left(\nabla^2\right)\Lambda_2\left(\nabla^2\right) = \prod_{j=1}^{4}\left(\nabla^2 + l_j^4\right),$$

$$\Lambda_{l+3,j+3}\left(\nabla^2\right) = \Lambda_2\left(\nabla^2\right)\left(\nabla^2 + l_5^2\right) = \prod_{j=3}^{5}\left(\nabla^2 + l_j^4\right),$$

$$\Lambda_{77}\left(\nabla^2\right) = \Lambda_1\left(\nabla^2\right)\Lambda_{pq}\left(\nabla^2\right) = 0,$$

$$l = 1,2,3, \quad p,q = 1,2,....,7\ p \neq q.$$

Using Eqs (5.9), (5.10) in Eq.(5.18), we obtain

$$\begin{aligned}\boldsymbol{\Phi'} = &\left[b_2\left(\gamma'_3\nabla^2 + \mu'_1\right)\Lambda_1\left(\nabla^2\right)\boldsymbol{J} + \eta'\left(\nabla^2\right)\mathrm{grad\,div}\right]\boldsymbol{F'} - \\ &+K'_3 b_2\Lambda_1\left(\nabla^2\right)\mathrm{curl}\,\boldsymbol{F''} - b_1 m_0\Lambda_2\left(\nabla^2\right)\mathrm{grad}\,F_0.\end{aligned} \tag{5.23}$$

From Eq.(5.21), we have

$$\begin{aligned}\boldsymbol{\Phi''} = &-b_2K'_3\left(\nabla^2 + l_5^2\right)\mathrm{curl}\,\boldsymbol{F'} + \\ &+\left\{b_2\left(\nabla^2 + l_5^2\right)\left[(\mu'_3 + K'_3)\nabla^2 + \rho\omega^2\right]\boldsymbol{J}\,\eta''\left(\nabla^2\right)\mathrm{grad\,div}\right\}\boldsymbol{F''}\end{aligned} \tag{5.24}$$

where

$$\eta'\left(\nabla^2\right) = -b_1b_2\left\{\left(k_0\nabla^2 + a_0\right)\left[(\lambda'_3 + \mu'_3)\left(\gamma'_3\nabla^2 + \mu'_1\right) - K'^2_3\right] + c'^2_1 m_0\left(\gamma'_3\nabla^2 + \mu'_1\right)\right\}, \tag{5.25}$$

$$\eta''\left(\nabla^2\right) = -\frac{b_2}{\gamma_0}\left\{(\alpha'_3 + \beta'_3)\left[(\mu'_3 + K'_3)\nabla^2 + \rho\omega^2\right] - K'^2_3\right\} \tag{5.26}$$

and $\boldsymbol{J} = \left\|\delta_{kl}\right\|_{3\times 3}$ represents the unit matrix.
From Eqs (5.10), (5.23) and (5.24), we obtain

$$\tilde{\boldsymbol{\Phi}}(x) = \boldsymbol{N}^{\boldsymbol{T}}(D_x)\boldsymbol{F}(x) \tag{5.27}$$

where $\quad N(\boldsymbol{D_x}) = \left\|N_{pq}(\boldsymbol{D_x})\right\|_{7\times 7},$

$$N_{lj}(\boldsymbol{D_x}) = b_2\left(\gamma'_3\nabla^2 + \mu'_1\right)\Lambda_1\left(\nabla^2\right)\delta_{lj} + \eta'\left(\nabla^2\right)\frac{\partial^2}{\partial x_l \partial x_j},$$

$$N_{l,j+3}(\boldsymbol{D_x}) = -b_2 K'_3 \left(\nabla^2 + l_5^2\right)\sum_{r=1}^{3} \epsilon_{lrj} \frac{\partial}{\partial x_r} N_{l7}(\boldsymbol{D_x}) = b_1 c'^2_1 \frac{\partial}{\partial x_l},$$

$$N_{l+3,j}(\boldsymbol{D_x}) = -b_2 K'_3 \Lambda_1\left(\nabla^2\right)\sum_{r=1}^{3} \epsilon_{lrj} \frac{\partial}{\partial x_r},$$

$$N_{l+3,j+3}(\boldsymbol{D_x}) = b_2\left(\nabla^2 + l_5^2\right)\left[(\mu_{3'} + K'_3)\nabla^2 + \rho\omega^2\right]\delta_{lj} + \eta''\left(\nabla^2\right)\frac{\partial^2}{\partial x_l \partial x_j},$$

$$N_{l+3,7}(\boldsymbol{D_x}) = N_{7,j+3}(\boldsymbol{D_x}) = 0\, N_{7l}(D_x) = -b_1 m_0 \Lambda_2\left(\nabla^2\right)\frac{\partial}{\partial x_l},$$

$$N_{77}(D_x) = b_1\left[\mu_0 \nabla^2 + \rho\omega^2\right] l, \quad j = 1,2,3.$$

Equation (5.22) indicates that $\boldsymbol{\Lambda}(\boldsymbol{U}) = \boldsymbol{N^T M^T U}$ based on Eqs (5.4) and (5.27). It is evident that $\boldsymbol{N^T M^T} = \boldsymbol{\Lambda}$ and therefore

$$\boldsymbol{M}(\boldsymbol{D_x})\boldsymbol{N}(\boldsymbol{D_x}) = \boldsymbol{\Lambda}\left(\nabla^2\right). \tag{5.28}$$

Let us consider $l_p^2 \neq l_q^2$, where $p,q = 1,2,...,5$ and $p \neq q$.
Also, consider

$$\boldsymbol{Z}(\boldsymbol{x}) = \|Z_{pq}(x)\|_{7\times 7}\, Z_{ll}(x) = \sum_{j=1}^{4} q_{1j}\gamma_j(x),$$

$$Z_{l+3,j+3}(x) = \sum_{j=3}^{5} q_{2j}\gamma_j(x) Z_{77}(x) = \sum_{j=1}^{2} q_{3j}\gamma_j(x),$$

$$Z_{pq}(x) = 0, \quad l = 1,2,3, \quad p,q = 1,2,.....,7\; p \neq q,$$

where
$$\gamma_p(x) = -\frac{1}{4\pi|(x)|} e^{\iota l_p |(x)|}, \quad p = 1,2,....,5$$

$$q_{1j} = \prod_{l=1, l\neq j}^{4} (l_l^2 - l_j^2)^{-1}, \quad j = 1,2,3,4$$

$$q_{2p} = \prod_{l=3, l\neq j}^{5} (l_l^2 - l_j^2)^{-1}, \quad p = 3,4,5$$

$$q_{31} = -q_{32} = \frac{1}{\left(l_2^2 - l_1^2\right)}.$$

Lemma: **Z** represents the basic matrix of operator $\boldsymbol{\Lambda}\left(\nabla^2\right)$ meaning that

$$\boldsymbol{\Lambda}\left(\nabla^2\right)\boldsymbol{Z}(\boldsymbol{x}) = \delta(x)\boldsymbol{I}. \tag{5.29}$$

Proof: It is sufficient to show

$$\Lambda_1\left(\nabla^2\right)\Lambda_2\left(\nabla^2\right)Z_{11}(x) = \delta(x), \tag{5.30}$$

$$\Lambda_2\left(\nabla^2\right)\left(\nabla^2 + l_5^2\right)Z_{44}(x) = \delta(x). \tag{5.31}$$

Accounting for the Equitables

$$\left(\nabla^2 + l_5^2\right)\gamma_j(x) = \delta(x) + \left(l_l^2 - l_j^2\right)\gamma_j(x)l, \quad j = 1,2$$

we have

$$\begin{aligned}
\Lambda_1\left(\nabla^2\right)Z_{77}(x) &= \left(\nabla^2 + l_1^2\right)\left(\nabla^2 + l_2^2\right)\sum_{j=1}^{2} q_{3j}\gamma_j(x) = \\
&= \frac{1}{\left(l_2^2 - l_1^2\right)}\left(\nabla^2 + l_1^2\right)\left(\nabla^2 + l_2^2\right)\left[\gamma_1(x) - \gamma_2(x)\right] = \\
&= \frac{1}{\left(l_2^2 - l_1^2\right)}\left(\nabla^2 + l_1^2\right)\left[\delta(x) + \left(l_2^2 - l_1^2\right)\gamma_1(x) - \delta(x)\right] = \delta(x).
\end{aligned}$$

There is a similar proof for Eqs (5.30) and (5.31).
We can define

$$\boldsymbol{G}(x)\boldsymbol{N}(x) = \boldsymbol{Z}(x). \tag{5.32}$$

Using Eq.(5.29) in Eqs (5.28) and (5.32), we obtain

$$\boldsymbol{M}(\boldsymbol{D_x})\boldsymbol{G}(x) = \boldsymbol{M}(D_x)\boldsymbol{N}(D_x)\boldsymbol{Z}(x) = \boldsymbol{\Lambda}\left(\nabla^2\right)\boldsymbol{Z}(x) = \delta(x)\boldsymbol{I}. \tag{5.33}$$

Thus, we simplified the subsequent theorem.

Theorem: If condition (4.4) is satisfied, the fundamental solution of Eqs (4.1)-(4.3) is the matrix $\boldsymbol{G}(x)$, given by Eq.(5.32).

***Corollary** 1*: For all $x \in E^3$, except the origin, each column of the matrix G(x) is the solution to Eqs (4.1)-(4.3).

***Corollary** 2*: $\boldsymbol{G}(x)$ can be

$$G_{pl}(x) = N_{pl}(\boldsymbol{D_x})Y_{11}(x),$$

$$G_{p,l+3}(x) = N_{p,l+3}(\boldsymbol{D}_x)Y_{44}(x),$$

$$G_{p7}(x) = N_{p7}(\boldsymbol{D}_x)Y_{77}(x), \quad l = 1,2,3, \quad p = 1,2,...,7 .$$

***Corollary** 3*: Equations (5.1)-(5.3) with the help of Eq.(4.4) become

$$(\mu'_3 + K'_3)\nabla^2 \boldsymbol{u}_1 + (\lambda'_3 + \mu'_3)\,\mathrm{grad\,div}\,\boldsymbol{u}_1 = 0, \tag{5.34}$$

$$\gamma'_3\nabla^2\phi_1 + (\alpha'_3 + \beta'_3)\,\mathrm{grad\,div}\,\phi_1 = 0, \tag{5.35}$$

$$k_1\nabla^2\theta = 0, \tag{5.36}$$

we obtain a matrix

$$\boldsymbol{\Gamma}(x) = \|\Gamma_{pq}(x)\|_{7\times7},$$

where
$$\Gamma_{lj}(x) = \lambda\frac{\delta_{lj}}{|x|} + \mu\frac{x_l x_j}{|x|^3},$$

$$\Gamma_{l+3,j+3}(x) = \lambda'\frac{\delta_{lj}}{|x|} + \mu'\frac{x_l x_j}{|x|^3}\Gamma_{77}(x) = -\frac{1}{4\pi k_0|x|},$$

$$\Gamma_{lr}(x) = \Gamma_{rl}(x) = \Gamma_{l+3,7}(x) = \Gamma_{7,l+3}(x) = 0,$$

$$\lambda = -\frac{\lambda'_3 + 3\mu'_3 + 2K'_3}{8\pi(\mu'_3 + K'_3)(\lambda'_3 + 2\mu'_3 + K'_3)}\mu = -\frac{\lambda'_3 + \mu'_3}{8\pi(\mu'_3 + K'_3)(\lambda'_3 + 2\mu'_3 + K'_3)},$$

$$\lambda = -\frac{\alpha'_3 + 2\gamma'_3 + \beta'_3}{8\pi\gamma'_3(\alpha'_3 + \beta'_3 + \gamma'_3)}\mu' = -\frac{\alpha'_3 + \beta'_3}{8\pi\lambda'_3(\alpha'_3 + \beta'_3 + \gamma'_3)}, \quad l,j = 1,2,3\, r = 1,2,3,4,5,6 .$$

6. Special cases

Case 1. If Eqs (4.1)-(4.3) ignore the impact of couple stress, micropolarity, and viscoelasticity, the equations reduce to

$$\mu_2\nabla^2\boldsymbol{u} + (\lambda_2 + \mu_2)\nabla(\nabla\cdot\boldsymbol{u}) - \vartheta_1\nabla T - \vartheta_2\nabla\dot{T} = \ddot{\boldsymbol{u}},$$

$$a_0\theta + m_0\,\mathrm{div}\,\boldsymbol{u}_1 + k_1\nabla^2\theta = 0,$$

which is comparable to what Hetnarski [28] found.

Case 2. Neglecting the impact of couple stress and viscoelastic in Eqs (4.1)-(4.3), the equations reduce to

$$\left(\mu_2+K_2\right)\nabla^2\boldsymbol{u}+\left(\lambda_2+\mu_2\right)\nabla\left(\nabla\cdot\boldsymbol{u}\right)+K_2\nabla\times\phi-\vartheta_1\nabla T-\vartheta_2\nabla\dot{T}=\ddot{\boldsymbol{u}},$$

$$a_0\theta+m_0\operatorname{div}\boldsymbol{u}_1+k_1\nabla^2\theta=0,$$

which is comparable to what Svanadze [30] found.

Case 3. Neglecting the effect of micropolar and viscoelastic in Eqs (4.1) -(4.3), the equations reduce to

$$\mu_2\nabla^2\boldsymbol{u}+\left(\lambda_2+\mu_2\right)\nabla\left(\nabla\cdot\boldsymbol{u}\right)+K_2\nabla\times\phi-\vartheta_1\nabla T-\vartheta_2\nabla\dot{T}=\ddot{\boldsymbol{u}},$$

$$a_0\theta+m_0\operatorname{div}\boldsymbol{u}_1+k_1\nabla^2\theta=0,$$

which is comparable to what Kumar *et al.* [34].

7. Particular cases

Case 1. Taking, $k=1$, we obtain

$$c_1^2=\vartheta_1,\quad c_3^2=\left(\frac{\vartheta_1}{\tau_3}+\frac{\vartheta_1}{\tau_2}\frac{\partial}{\partial t}\right),\quad c_4^2=\left(1+n_1\frac{\partial}{\partial t}\right).$$

These numbers allow us to determine the different wave speeds for the LS-model.

Case2. Putting $k=2$, we obtain

$$c_1^2=\left(\vartheta_1+\vartheta_2\frac{\partial}{\partial t}\right),\quad c_3^2=\frac{\vartheta_1}{\tau_3},\quad c_4^2=1.$$

These values allow us to determine the different wave speeds for the GL-model.

Case 3. Taking $\alpha_2^*=\beta_2^*=\gamma_2^*=\alpha_2=\beta_2=\gamma_2=0$ in Eqs (2.11)-(2.12.), then, which is similar to the results obtained by Kumar [42].

i. Taking $\alpha_2^*=\beta_2^*=\gamma_2^*=\alpha_2=\beta_2=\gamma_2=0$ in Eqs (2.11)-(2.12) and applying **case 1,** we find results similar to micropolar viscoelastic generalized thermoelastic solid for LS- model.
ii. Taking $\alpha_2^*=\beta_2^*=\gamma_2^*=\alpha_2=\beta_2=\gamma_2=0$ in Eqs (2.11)-(2.12)and applying **case 2**, then, we find results similar to micropolar viscoelastic generalized thermoelastic solid for GL- model.

Case 4. Taking $\lambda_2^*=\mu_2^*=\alpha_2^*=\beta_2^*=\gamma_2^*=0$ in Eqs (2.11)-(2.12), then, this is comparable to the results as obtained by Kumar *et al.* [34].

i. Taking $\lambda_2^*=\mu_2^*=\alpha_2^*=\beta_2^*=\gamma_2^*=0$ in Eq.(2.11)-(2.12) and applying **case 1**,we find results similar to micropolar couple stress generalized thermoelastic solid for LS- model.
ii. Taking $\lambda_2^*=\mu_2^*=\alpha_2^*=\beta_2^*=\gamma_2^*=0$ in Eq.(2.11)-(2.12) and applying **case 2**, we find results similar to micropolar couple stress generalized thermoelastic solid for GL- model.

Case 5. Taking $\lambda_2^* = \mu_2^* = \alpha_2^* = \beta_2^* = \gamma_2^* = \alpha_2 = \beta_2 = \gamma_2 = 0$ in Eqs (2.11)-(2.12), then, which is similar as obtained by Singh [43].

i. Suppose $\lambda_2^* = \mu_2^* = \alpha_2^* = \beta_2^* = \gamma_2^* = \alpha_2 = \beta_2 = \gamma_2 = 0$ in Eqs (2.11)-(2.12) an applying **case 1**, then, we find results similar to isotropic thermoelastic solid for LS- model.

ii. Suppose $\lambda_2^* = \mu_2^* = \alpha_2^* = \beta_2^* = \gamma_2^* = \alpha_2 = \beta_2 = \gamma_2 = 0$ in Eqs (2.11)-(2.12) and applying **case 2**, then, we find the similar results for isotropic thermoelastic solid for GL- model.

8. Numerical results

With the aid of MATLAB software, a graphical representation has been created to visually represent the effects of couple stress, viscous and micropolar on the phase velocity, specific loss, attenuation coefficients, and penetration depth with wave number under the LS- and GL-models. The comparison of different theories are discussed. Magnesium is taken for computational purpose as an example, for which the material constants are taken from Kumar [40] and Kumar *et al.* [41] at *293 K* which are given below.

Table 1. Numerical value of the parameter.

keynote	value	keynote	value	keynote	value
λ_2	$9.4\times10^{10}\,Nm^{-2}$	λ_2^*	$500\times10^{6}\,Nm^{-2}$	β_2	$2.48\times10^{-5}\,N$
K_2	$1.0\times10^{10}\,Nm^{-2}$	α_2^*	$5.27\times10^{3}\,N$	K_t	$1.7\times10^{6}\,jKm^{-1}s^{-1}K^{-1}$
j	$0.2\times10^{-19}\,m^2$	γ_2^*	$3.17\times10^{-3}\,N$	t_1	1.04
α_2	$2.33\times10^{5}\,N$	μ_2	$4.0\times10^{10}\,Nm^{-2}$	μ_2^*	$0.1\times10^{9}\,Nm^{-2}$
ϑ	$0.9\times10^{-17}\,N$	ρ	$1.73\times10^{3}\,Kgm^{-3}$	β_2^*	$4.0\times10^{6}\,N$
t_0	1.03	γ_1	$0.779\times10^{-9}\,N$	K_2^*	$0.5\times10^{10}\,N$

8.1. Phase velocity

Three-dimensional (3D) graphs impacted by couple stress, viscous and micropolar on phase velocities are displayed in Figs 1-4. These graphs show how the wave distributions of fundamental physical properties are affected by wave number and distance. The steady state states that as the wave number increases, all wave propagation varies and approaches a finite depth.

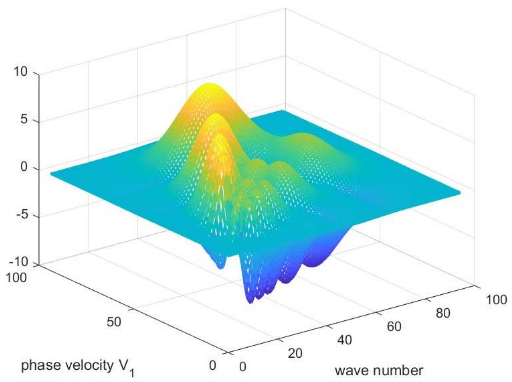

Fig.1. Phase velocity V_1 versus wave number.

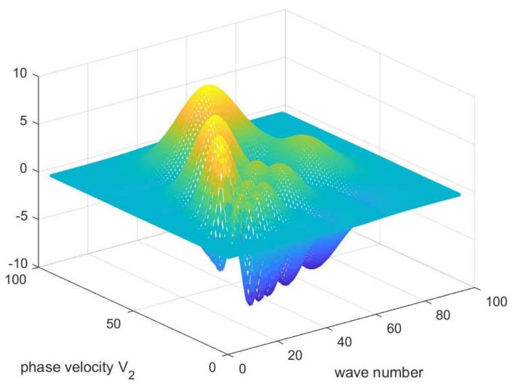

Fig.2. Phase velocity V_2 versus wave number.

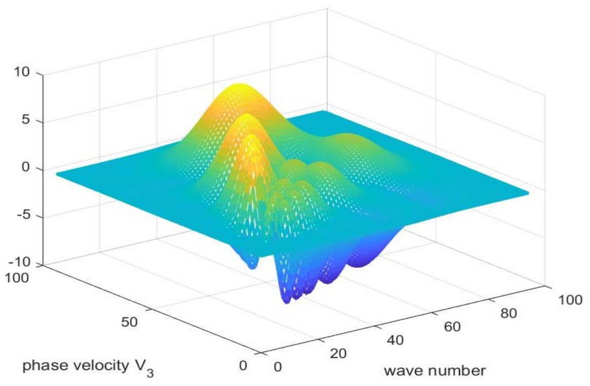

Fig.3. Phase velocity V_3 versus wave number.

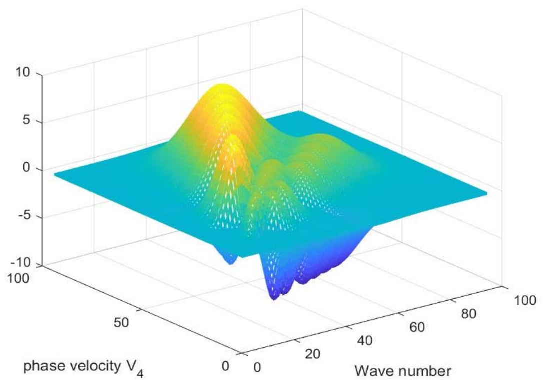

Fig.4. Phase velocity V_4 versus wave number.

8.2. Attenuation coefficients

The effects of viscous, couple stress and microploar on the attenuation coefficients $Q_s\left(s=1,2,3,4\right)$ in relation to wave number are shown in Figs 5-8. The solution is presented graphically with harmonic waves propagating through the medium in the *x*-direction and each curve's amplitude decreasing as the medium depth increases.

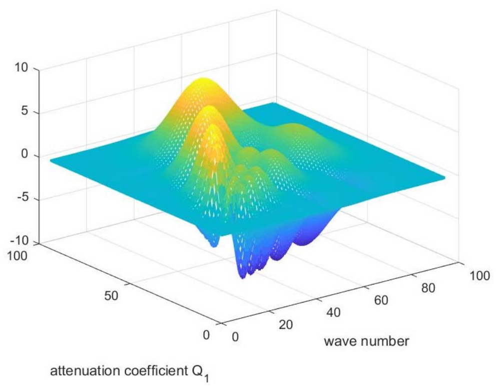

Fig.5. Attenuation coefficient Q_1 versus wave number.

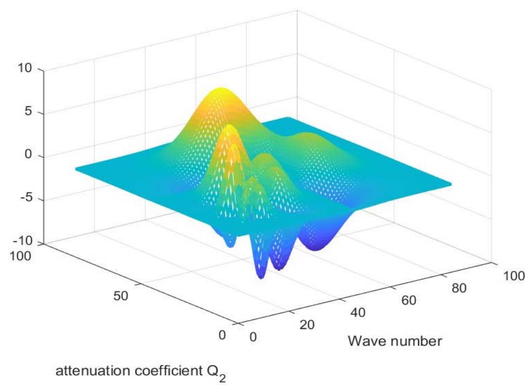

Fig.6. Attenuation coefficient Q_2 versus wave number.

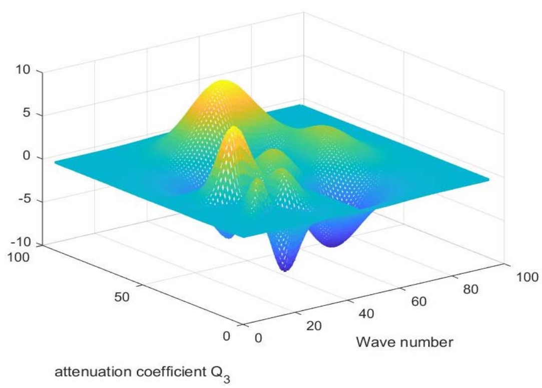

Fig.7. Attenuation coefficient Q_3 versus wave number.

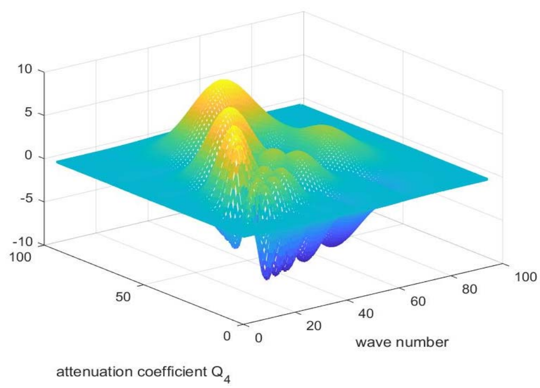

Fig.8. Attenuation coefficient Q_4 versus wave number.

8.3. Specific loss

The effect of viscous, couple stress and microploar on the specific loss $R_s\left(s=1,2,3,4\right)$ in relation to wave number is shown in Figs. 9-12. Three-dimensional figures show the impact of couple stress and viscosity.

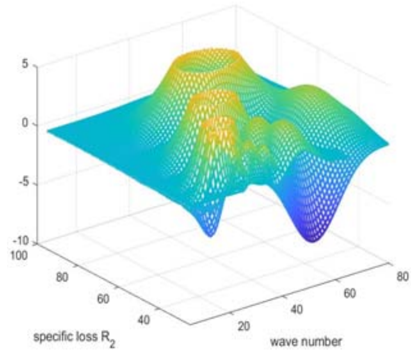

Fig.9. Specific loss R_1 versus wave number.

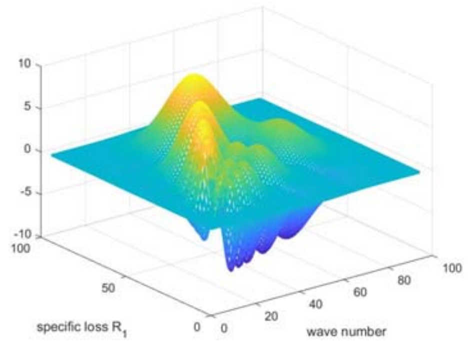

Fig.10. Specific loss R_2 versus wave number.

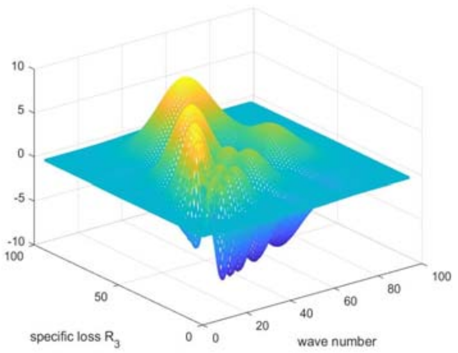

Fig.11. Specific loss R_3 versus wave number.

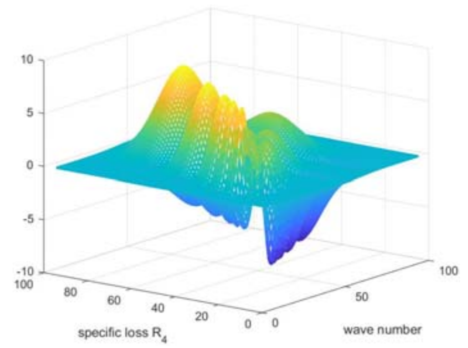

Fig.12. Specific loss R_4 versus wave number.

8.4. Penetration depth

The influence of viscous, couple stress and micropolar on the penetration depth $L_s\left(s=1,2,3,4\right)$ in relation to wave number is shown in Figs. 13-16. The couple stress and viscosity have a significant impact on the penetration depth L_s as the figures below illustrate.

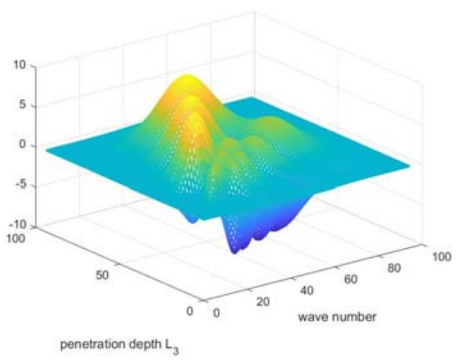

Fig.13. Penetration depth P_1 versus wave number.

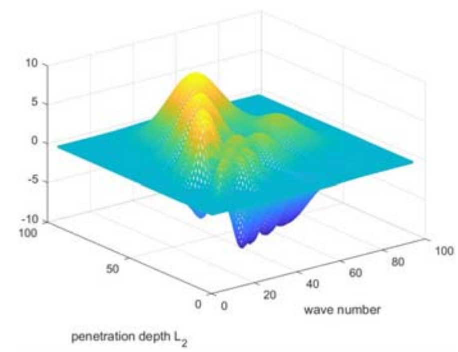

Fig.14. Penetration depth P_2 versus wave number.

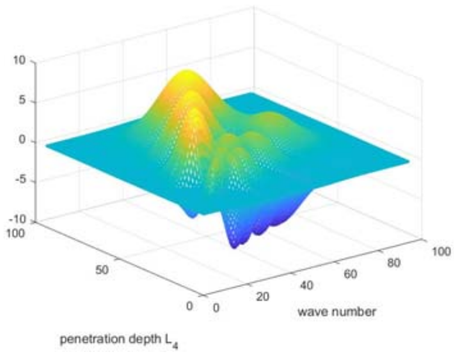

Fig.15. Penetration depth P_3 versus wave number.

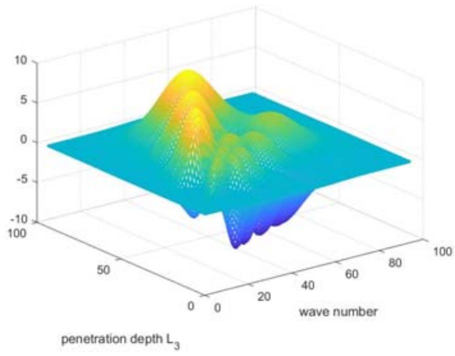

Fig.16. Penetration depth P_4 versus wave number.

9. Conclusion

We investigated the impact of viscous parameters, couple stress and micropolar on plane wave propagation in a generalised thermoelastic solid medium with couple stress micropolar viscoelastic behaviour. The following are the main effects of this issue:

1. It was discovered that there are four distinct waves, each propagating at a distinct speed. In addition, a numerical analysis and graphical representation of the penetration depth, specific loss, attenuation coefficients, and phase velocity with respect to wave number are provided. The graphical discussion is based on the different theories of thermoelasticity.
2. The properties of the waves are discovered to be influenced by couple stress , viscous and micropolar parameters under LS and GL model of the given medium.
3. In LS- model, couple stress, viscous and micropolar parameters have more influence on wave characteristics Q_1, Q_3, Q_4, R_3 L_1 and L_3. Whereas in case of GL model remaining characteristics V_1, V_2, V_4, Q_2, L_2, L_4, R_1, R_2 and R_4 are influenced more by these parameters.
4. Additionally, we noted that while couple stress had no discernible effect on phase velocity V_3, it did have a significant impact on penetration depth.
5. The 3-D boundary value problems of couple stress micropolar thermoviscous elastic solid can be examined with the aid of theoretical and numerical analysis of $M(x)$ of the Eqs (4.1)-(4.5).
6. The problems discussed in the paper can be extended to include the impact of polar and spherical coordinates, orthotropy and higher order symmetry.

Acknowledgement

Sonam, one of the authors, is appreciative of the financial support in the form of a Senior Research fellowship from UGC (16-9(June 2019)/2019(NET/CSIR)).

Nomenclature

K_t , K_2 – material constants

R – macrorotation vector

T – temperature

t_0 , t_1 – relaxation time

α_2 , α_2^* – material constants

β_2 , β_2^* – material constants

γ_2 , γ_2^* – material constants

δ_{ij} – Kronecker's delta

ϵ_{ij} – alternating tensor

η – entropy

λ_2^* – material constant

μ_2^* – material constant

ρ – density

ϑ – material constant

ϕ – microrotation vector

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