In this paper, analytical solutions are presented for temperature and thermal behavior of a thermosensitive multilayered annular disc due to point heat source. Convective heating is applied to both the innermost and outermost layers. The nonlinearity of the thermal diffusivity equation is dealt using Kirchhoff’s transformation technique. A finite integral transform in the form of Bessel’s functions is used to deal with the radial variable $r$. Fourier transform and angular eigen functions are also used to solve the thermal diffusivity equation. Deflection, resultant forces, shearing forces, resultant moments and thermal stresses are obtained. A mathematical representation is formulated for a 3-layered disc, with the inner, middle and outer layers composed of copper, zinc and aluminum respectively. The results are depicted graphically.

**Key words:** multilayered annular disc, thermosensitive, heat conduction, instantaneous point heat source, deflection, stresses.

1. Introduction

Multilayered configurations like composite and sandwich plates, circular discs, comprise multiple layers that offer superior structural and thermal characteristics. These structures find extensive applications in aviation, civil and offshore engineering. The careful consideration of mechanical performance under temperature variations is crucial in the production of multiple layered structures. Temperature changes cannot just cause significant inner pressures but may impact the characteristics of the materials within these structures. Due to increase in the use of mechanical structures in high temperature environment in the last three decades, the study of thermo-mechanical behavior of different materials with properties that vary with temperature has received attention.

Noda [1] discussed the influence of physical characteristics affected by temperature on thermal behavior of different solids. Olcer [2] presented a thorough examination of temperature profile by considering finite-length...

Jangid and Mukhopadhyay [23] presented an alternative solution for a initial-value and boundary-value problem. Etkin [24] elucidated the thermal-impulse's physical significance through the concept of entropy. Srinivas et al. [25] carried out thermoelastic analysis by taking rectangular-parallelepiped subjected to convection and temperature dependent characteristics. Razavi et al. [26], Balci and Akpinar [27], Bikram and Kedar [28], Etkin [29], Su et al. [30] solved various problems of steady state and un-steady state temperature profile and analyzed the corresponding thermal stresses. In order to ascertain the expression of temperature and stresses, Lamba [31] examined the behaviour of fractional time derivative in temperature-sensitive FG cylinders. Recently, in a thermo-diffusive medium, Yadav et al. [32] successfully established a significant memory response. Also the related work is reflected in [33, 34].

In this work the authors try to investigate the influence of point heating on temperature and thermal profile of a thermally sensitive multilayered annular circular disc. The $k$ layered disc is defined over $r_{i-1} < r < r_i$, $0 < \theta < 2\pi$, $0 < z < h$. Heat conduction equation (HCE) is solved by integral-transform method and thermoelastic behavior is analyzed. Graphical analysis is carried out for a 3-layered disc.

2. Heat conduction equation and its solution

Figure 1 gives the depiction of the layered circular disc in a geometric form.

![Fig.1. Multilayered disc.](image)

The unsteady HCE with internal heat generation of a multilayered annular circular disc is [9]:

$$\frac{I}{r} \frac{\partial}{\partial r} \left( r \lambda_i(T_i) \frac{\partial T_i}{\partial r} \right) + \frac{I}{r^2} \frac{\partial}{\partial \theta} \left( \lambda_i(T_i) \frac{\partial T_i}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \lambda_i(T_i) \frac{\partial T_i}{\partial z} \right) + \rho_i c_i(T_i) \frac{\partial T_i}{\partial t} = Q(r, \theta, z, t)$$

(2.1)
where $\lambda_i(T_i)$, $C_i(T_i)$ are respectively, the temperature dependent thermal conductivity, specific heat capacity of the $i^{th}$ layer, $\rho_i$ is the density of the $i^{th}$ layer, $Q(r,\theta,z,t)$ is the internal heat generation and $i=1,2,3,\ldots,k$.

Following [9], the initial, boundary, inner and outer surface, interface, periodic boundary conditions are given in Eqs (2.2) to (2.7).

\begin{align}
T_i &= 0, \quad \text{at} \quad t=0, \quad (2.2) \\
\lambda_i(T_i) \frac{\partial T_i}{\partial r} + h_0 T_i &= 0, \quad \text{at} \quad r=r_0, \quad (2.3) \\
\lambda_k(T_k) \frac{\partial T_k}{\partial r} + h_k T_k &= f_k(z,\theta,t), \quad \text{at} \quad r=r_k, \quad (2.4) \\
T_i(r_{i-1},\theta,z,t) &= T_{i-1}(r_{i-1},\theta,z,t), \quad (2.5) \\
\lambda_i(T_i) \frac{\partial T_i}{\partial r} \bigg|_{r=r_{i-1}} = \lambda_{i-1}(T_{i-1}) \frac{\partial T_{i-1}}{\partial r} \bigg|_{r=r_{i-1}}, \quad (2.6) \\
T_i |_{\theta=0} = T_i |_{\theta=2\pi}, \quad \lambda_i(T_i) \frac{\partial T_i}{\partial \theta} \bigg|_{\theta=0} = \lambda_i(T_i) \frac{\partial T_i}{\partial \theta} \bigg|_{\theta=2\pi}, \quad (2.7)
\end{align}

We use following dimensionless parameters.

\begin{align}
T_i &= \frac{T_i}{T_0}, \quad r^* = \frac{r}{h}, \quad \theta^* = \frac{\theta}{2\pi}, \quad z^* = \frac{z}{h}, \quad t^* = \frac{\kappa_i t}{h^2}, \quad \rho_i^* = \frac{\rho_i}{\rho_j}, \quad (2.8) \\
(r^*_i,h^*_i) &= \left(\frac{r_i h}{h}\right), \quad E_i^* = \frac{E_i}{E_j}, \quad a^* = \frac{a h^2}{\kappa_j}, \quad \rho_j^* = \frac{\rho_j h^2}{\kappa_j}, \quad j=1,2,
\end{align}

where $T_0$ is the ambient temperature, $h$ is the thickness of the disc, $\kappa_i = \lambda_i / (C_i \rho_i)$, is the thermal diffusivity of the inner layer, $\lambda_i, C_i, \rho_i$ are heat transfer properties, density, $E_i$ is the Young’s modulus, $a, \rho_j^*$ are the frequency.

The temperature dependent material properties $\lambda_i(T_i)$, $C_i(T_i)$, and heat flow $f_k(z,\theta,t)$ are taken as [4, 5, 13]

\begin{align}
\lambda_i(T_i) &= \lambda_i(T_i^*), \quad C_i(T_i) = C_i^*(T_i^*), \\
f_k(z,\theta,t) &= f_0 f^*_k(z^*, \theta^*, t^*), \quad Q(r,\theta,z,t) = Q_0 Q^*(r^*, \theta^*, z^*, t^*), \quad (2.9)
\end{align}

where $\lambda_j, C_j$ have dimensions, $f_0, Q_0$ are the strength of the heat flow having relevant dimensions, and $\lambda_i^*(T_i^*), C_i^*(T_i^*)$ are the dimensionless quantities, which are functions that describe the dependence of
these characteristics on dimensionless temperature, \( f_\ast(z^\ast, \Theta^\ast, t^\ast) \), \( Q^\ast(r^\ast, \Theta^\ast, z^\ast, t^\ast) \) are the dimensionless functions which describe the space distribution of the heat flow. Using Eqs (2.8-2.9), Eqs (2.1-2.7) reduces to the following dimensionless form (ignoring asterisks for convenience).

\[
\frac{l}{r} \frac{\partial}{\partial r} \left( r \lambda_i(T_i) \frac{\partial T_i}{\partial r} \right) + \frac{l}{4\pi^2 r^2} \frac{\partial}{\partial \theta} \left( \lambda_i(T_i) \frac{\partial T_i}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \lambda_i(T_i) \frac{\partial T_i}{\partial z} \right) + P_0 Q(r, \Theta, z, t) = \rho_i C_i(T_i) \frac{\partial T_i}{\partial t},
\]

(2.10)

The initial, boundary, inner and outer surface, interface, periodic boundary conditions are given in Eqs (2.11) to (2.16)

\[
T_i = 0, \text{ at } t = 0,
\]

(2.11)

\[
\lambda_i(T_i) \frac{\partial T_i}{\partial r} + B_i T_i = 0, \text{ at } r = \Phi_i,
\]

(2.12)

\[
\lambda_k(T_k) \frac{\partial T_k}{\partial r} + B_i T_k = K_i f_k(z, \Theta, t), \text{ at } r = \Phi_k,
\]

(2.13)

\[
T_i(r_{i-1}, \Theta, z, t) = T_{i-1}(r_{i-1}, \Theta, z, t),
\]

(2.14)

\[
\lambda_i(T_i) \frac{\partial T_i}{\partial r} \bigg|_{r = r_{i-1}} = \lambda_{i-1}(T_{i-1}) \frac{\partial T_{i-1}}{\partial r} \bigg|_{r = r_{i-1}},
\]

(2.15)

\[
T_i \bigg|_{\Theta = 0} = T_i \bigg|_{\Theta = 1},
\]

(2.16)

Here \( P_0 = \frac{Q_0 h^2}{\lambda_j T_0} \) and \( K_i = \frac{f_0 h}{\lambda_j T_0} \) are respectively the dimensionless Pomerantsev reference number and dimensionless Kirpichev reference number, \( B_i = \frac{h_0 h}{\lambda_j} \), \( B_i = \frac{h_0 h}{\lambda_j} \) are the Biot criteria and \( \Phi_i = (r_0 / h), \Phi_2 = (r_k / h) \).

Introducing Kirchhoff’s variable transformation [4, 5, 13]

\[
\Theta_i(T_i) = \int_0^{T_i} \lambda_i(T_i) d T_i,
\]

(2.17)
Thermal stresses associated with a thermosensitive multilayered...

and considering the material with simple thermal nonlinearity (that is \( [C_i(T_i) / \lambda_i(T_i)] = I \)), Eqs (2.10) to (2.16) become:

\[
\frac{\partial^2 \Theta_i}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta_i}{\partial r} + \frac{1}{4\pi r^2} \frac{\partial^2 \Theta_i}{\partial \theta^2} + \frac{\partial^2 \Theta_i}{\partial z^2} + P_i Q(r, \Theta_i(z, t)) = \rho_i \frac{\partial \Theta_i}{\partial t}.
\]  

(2.18)

The initial, boundary, inner and outer surface, interface, periodic boundary conditions are given in Eqs (2.19)-(2.24).

\[
\Theta_i = 0, \quad \text{at} \quad t = 0,
\]  

(2.19)

\[
\frac{\partial \Theta_i}{\partial r} + B_i \Theta_i = 0, \quad \text{at} \quad r = \Phi_1,
\]  

(2.20)

\[
\frac{\partial \Theta_k}{\partial r} + B_i \Theta_k = K_i f_k(z, \Theta, t), \quad \text{at} \quad r = \Phi_2,
\]  

(2.21)

\[
\Theta_i(r_{i-1}, \Theta, z, t) = \Theta_{i-1}(r_{i-1}, \Theta, z, t),
\]  

(2.22)

\[
\left. \frac{\partial \Theta_i}{\partial r} \right|_{r=r_{i-1}} = \left. \frac{\partial \Theta_{i-1}}{\partial r} \right|_{r=r_{i-1}},
\]  

(2.23)

\[
\Theta_i \big|_{\theta=0} = \Theta_i \big|_{\theta=l},
\]  

(2.24)

To solve HCE (2.18), let:

\[
f_k(\Theta, z, t) = \delta(\Theta - \Theta_0) \delta(z - z_0) \exp(at),
\]  

\[
Q(r, \Theta, z, t) = \delta(r - r_0) \delta(\Theta - \Theta_0) \delta(z - z_0) \delta(t).
\]  

Using Fourier transform on Eqs (2.18) and (2.24), we get:

\[
\frac{\partial^2 \bar{\Theta}_i}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\Theta}_i}{\partial r} + \frac{1}{4\pi^2 r^2} \frac{\partial^2 \bar{\Theta}_i}{\partial \Theta^2} - \psi_n^2 \bar{\Theta}_i + \bar{Q}(r, \Theta, \psi_n, t) = \rho_i \frac{\partial \bar{\Theta}_i}{\partial t}.
\]  

(2.25)

The conditions (2.19-2.23) become;

\[
\bar{\Theta}_i = 0, \quad \text{at} \quad t = 0,
\]  

(2.26)
\[
\frac{\partial \Theta_i}{\partial r} + Bi_i \Theta_i = 0, \quad \text{at } r = \Phi_i, \quad \text{(2.27)}
\]

\[
\frac{\partial \Theta_k}{\partial r} + Bi_2 \Theta_k = \overline{f}_k (\psi_n, \Theta, t), \quad \text{at } r = \Phi_2, \quad \text{(2.28)}
\]

\[
\Theta_i (n_i, \theta, \psi_n, t) = \overline{\Theta}_{i-i} (n_i, \theta, \psi_n, t), \quad \text{(2.29)}
\]

\[
\left. \frac{\partial \Theta_i}{\partial r} \right|_{r=\eta_i} = \left. \frac{\partial \Theta_{i-i}}{\partial r} \right|_{r=\eta_i},
\]

\[
\Theta_i \big|_{\theta=0} = \Theta_i \big|_{\theta=\pi}, \quad \text{(2.30)}
\]

where

\[
\psi_n = n \pi / \delta, \quad \overline{f}_k (\psi_n, \theta, t) = K_i z_0 \sin(n \pi z_0 / \delta) \delta(\theta - \theta_0) \exp(at),
\]

\[
\overline{Q}(r, \theta, z, t) = (z_0 / P_0) \sin(n \pi z_0 / \delta) \delta(r - r_0) \delta(\theta - \theta_0) \delta(t).
\]

Considering periodic conditions, \( \overline{\Theta}_i (r, \theta, t) \) is expanded as:

\[
\overline{\Theta}_i (r, \theta, t) = \overline{\Theta}_{i0} (r, t) + \sum_{m=1}^{\infty} \overline{\Theta}_{imc}(r, t) \cos(m\theta) + \sum_{m=1}^{\infty} \overline{\Theta}_{im}(r, t) \sin(m\theta). \quad \text{(2.31)}
\]

Similarly, the expression for heat supply is taken as:

\[
\overline{f}_k (\theta, t) = \overline{f}_{k0} (\theta, t) + \sum_{m=1}^{\infty} \overline{f}_{kmc}(\theta, t) \cos(m\theta) + \sum_{m=1}^{\infty} \overline{f}_{km}(\theta, t) \sin(m\theta). \quad \text{(2.32)}
\]

Using the orthogonality conditions along axial direction, the coefficients in Eq.(2.32) are obtained as:

\[
\overline{f}_{i0}(r, t) = \int_0^l \overline{f}_i (r, \theta, t) d\theta,
\]

\[
\overline{f}_{imc}(r, \theta, t) = \int_0^l \overline{f}_i (r, \theta, t) \cos(m\theta) d\theta,
\]

\[
\overline{f}_{ims}(r, \theta, t) = \int_0^l \overline{f}_i (r, \theta, t) \sin(m\theta) d\theta. \quad \text{(2.33)}
\]
Using Eqs (2.31) and (2.32) in Eqs (2.25) and (2.30), we get:

\[
\frac{\partial^2 \bar{\Theta}_{im}}{\partial r^2} + \frac{l}{r} \frac{\partial \bar{\Theta}_{im}}{\partial r} - \frac{m^2}{4\pi^2 r^2} \bar{\Theta}_{im} - \Psi_n^2 \bar{\Theta}_{im} + \bar{Q}_m(r,t) = \rho_i \frac{\partial \bar{\Theta}_{im}}{\partial t}.
\]  

(2.34)

The conditions (2.26-2.29) become

\[
\bar{\Theta}_{im}(t=0, r) = 0,
\]

(2.35)

\[
\frac{\partial \bar{\Theta}_{im}}{\partial r} + B_{ij} \bar{\Theta}_{im}(r = \Phi_j) = 0,
\]

(2.36)

\[
\frac{\partial \bar{\Theta}_{km}}{\partial r} + B_{ij} \bar{\Theta}_{km}(r = \Phi_2) = \bar{f}_{km}(t),
\]

(2.37)

\[
\bar{\Theta}_{im}(r_{i-1}, t) = \bar{\Theta}_{i-1,m}(r_{i-1}, t),
\]

(2.38)

where

\[
\bar{f}_{km}(t) = A_t \exp(\alpha t), A_t = K_i \theta_0 z_0 \sin(m \theta_0 / 2) \sin(n \pi z_0 / h),
\]

\[
\bar{Q}_m(r, \psi_a, t) = A_2 \delta(r - r_a) \delta(t), A_2 = (z_0 \theta_0 / P_0) \sin(n \pi z_0 / h) \sin(m \theta_0 / 2).
\]

Using integration and operating eqn. (2.34) by \[\int_{r_{i-1}}^{r_i} r S_{im}(r) \, dr\], we get [9]

\[
\int_{r_{i-1}}^{r_i} \left( \frac{\partial^2 S_{im}(r)}{\partial r^2} + \frac{l}{r} \frac{\partial S_{im}(r)}{\partial r} - \frac{m^2}{4\pi^2 r^2} S_{im}(r) - \Psi_n^2 S_{im}(r) \right) r \bar{\Theta}_{im} \, dr +
\]

\[
\left[ \frac{r S_{im}(r) \frac{\partial \bar{\Theta}_{im}}{\partial r} - r \bar{\Theta}_{im} \frac{\partial S_{im}(r)}{\partial r}}{r}ight]_{r_{i-1}}^{r_i} + \int_{r_{i-1}}^{r_i} \left( \bar{Q}_m(r, t) \right) r S_{im}(r) \, dr = \int_{r_{i-1}}^{r_i} \left( \rho_i \frac{\partial \bar{\Theta}_{im}}{\partial t} \right) r S_{im}(r) \, dr.
\]

(2.39)

\(S_{im}(r)\) in Eq.(2.39) is chosen so that it satisfies the following differential equation

\[
\frac{\partial^2 S_{im}(r)}{\partial r^2} + \frac{l}{r} \frac{\partial S_{im}(r)}{\partial r} + \left( -\frac{m^2}{4\pi^2 r^2} - \Psi_n^2 + \alpha_{im} \right) S_{im}(r) = 0.
\]

(2.40)

Subject to inner and outer surface, interface conditions
\[
\frac{dS_{\text{im}}}{dr} + B_{1}S_{\text{im}} = 0, \quad (2.41)
\]

\[
\frac{dS_{\text{km}}}{dr} + B_{2}S_{\text{km}} = 0, \quad (2.42)
\]

\[
S_{\text{im}}(r_{i-1}) = S_{i-1,m}(r_{i-1}), \quad (2.43)
\]

Solution of these eqns. in terms of Bessel’s function is expressed as:

\[
S_{\text{imp}}(\alpha_{\text{imp}} r) = a_{\text{imp}}J_{0}(\alpha_{\text{imp}} r) + b_{\text{imp}}Y_{0}(\alpha_{\text{imp}} r).
\]

Orthogonality condition

\[
\sum_{i=1}^{k} \int_{r_{i}}^{r_{i+1}} rS_{\text{imp}}(\alpha_{\text{imp}} r)S_{\text{imp}}(\alpha_{\text{imp}} r) dr = \begin{cases} 0; & p \neq q, \\ S_{\text{imp}}(\alpha_{\text{imp}}) ; & p = q, \end{cases} \quad (2.44)
\]

\[
\kappa_{i}\alpha_{\text{imp}}^{2} = \kappa_{i}\alpha_{\text{imp}}^{2}
\]

Using Eq.(2.40), Eq.(2.39) becomes:

\[
\left[ rS_{\text{imp}}(r) \frac{\partial \overline{\Theta}_{\text{im}}}{\partial r} - r \overline{\Theta}_{\text{im}} \frac{\partial rS_{\text{imp}}}{\partial r} \right]_{r_{i-1}}^{r_{i}} + \int_{r_{i}}^{r_{i+1}} (\overline{Q}_{\text{m}}(r,t)) rS_{\text{imp}}(r) dr = \frac{r_{i}}{r_{i-1}} \left( \rho_{i} \frac{\partial \overline{\Theta}_{\text{im}}}{\partial t} + \alpha_{\text{imp}}^{2} \overline{\Theta}_{\text{im}} \right) rS_{\text{imp}}(r) dr.
\]

Using Eq.(2.45), Eq.(2.46) becomes:

\[
\left[ rS_{\text{imp}}(r) \frac{\partial \overline{\Theta}_{\text{im}}}{\partial r} - r \overline{\Theta}_{\text{im}} \frac{\partial rS_{\text{imp}}}{\partial r} \right]_{r_{i-1}}^{r_{i}} + \int_{r_{i}}^{r_{i+1}} (\overline{Q}_{\text{m}}(r,t)) rS_{\text{imp}}(r) dr = \frac{r_{i}}{r_{i-1}} \left( \rho_{i} \frac{\partial \overline{\Theta}_{\text{im}}}{\partial t} + \frac{\kappa_{i}}{\kappa_{i}} \alpha_{\text{imp}}^{2} \overline{\Theta}_{\text{im}} \right) rS_{\text{imp}}(r) dr.
\]

After simplification, we get:
\[
\sum_{i=1}^{k} \lambda_i \left[ r S_{\text{imp}}(r) \frac{\partial \tilde{\Theta}_{\text{im}}}{\partial r} - r \tilde{\Theta}_{\text{im}} \frac{\partial S_{\text{imp}}(r)}{\partial r} \right]_{\eta_{i-1}}^{\eta_i} + \sum_{i=1}^{k} \int_{\eta_{i-1}}^{\eta_i} \lambda_i \tilde{\omega}_{\text{im}}(r,t) r S_{\text{imp}}(r) \, dr = 0.
\]  

(2.48)

We define

\[
\tilde{\Theta}_{\text{mp}}^\phi = \sum_{i=1}^{k} \int_{\eta_{i-1}}^{\eta_i} r S_{\text{imp}}(r) \tilde{\Theta}_{\text{im}} \, dr,
\]

(2.49)

\[
\tilde{\mathcal{Q}}_{\text{mp}}^\phi = \sum_{i=1}^{k} \int_{\eta_{i-1}}^{\eta_i} \lambda_i \tilde{\mathcal{Q}}_{\text{im}}(r,t) r R_{\text{mp}}(r) \, dr.
\]

Hence Eq.(2.48) becomes:

\[
\rho_i \frac{d \tilde{\Theta}_{\text{mp}}^\phi}{dt} + (\kappa_i / \kappa_i) \alpha_{\text{imp}}^2 \tilde{\Theta}_{\text{mp}}^\phi = \sum_{i=1}^{k} \lambda_i \left[ r S_{\text{imp}}(r) \frac{\partial \tilde{\Theta}_{\text{im}}}{\partial r} - r \tilde{\Theta}_{\text{im}} \frac{\partial S_{\text{imp}}(r)}{\partial r} \right]_{\eta_{i-1}}^{\eta_i} + \tilde{\Theta}_{\text{mp}}^\phi.
\]  

(2.50)

Applying the interface conditions (2.38) and (2.43), yields

\[
\frac{d \tilde{\Theta}_{\text{mp}}^\phi}{dt} + A_3 \tilde{\Theta}_{\text{mp}}^\phi = A_4 \exp(at) + A_5 \delta(t)
\]  

(2.51)

where

\[
A_3 = (\kappa_i / \rho_i \kappa_i) \alpha_{\text{imp}}^2, \quad A_4 = A_1 \alpha_{\text{k}} r_{\text{k}} / \rho_{\text{k}},
\]

\[
A_5 = A_2 F(t_i, \eta_{i-1}), \quad F(t_i, \eta_{i-1}) = \int_{\eta_{i-1}}^{\eta_i} [\delta(r-r_a) r S_i(r) \, dr]
\]

and

\[
\tilde{\Theta}_{\text{mp}}^\phi = 0, \text{ at } t=0.
\]  

(2.52)

Using Laplace transform (LT), LT inversion on Eqs (2.51), (2.52), we get:

\[
\tilde{\Theta}_{\text{mp}}^\phi = E_1[\exp(at) - \exp(-A_3t)] + A_5 \exp(-A_3t).
\]  

(2.53)

Here \( E_1 = (A_4 / A_3 + a) \).

The generalized Fourier series expansion of \( \tilde{\Theta}_{\text{im}}(r,t) \) is
\(\bar{\Theta}_{im}(r,t) = \sum_{p=1}^{\infty} c_{mp}(t)S_{imp}(r),\) \hspace{1cm} (2.54)

where

\[c_{mp}(t) = \left[ \sum_{l=1}^{k} \lambda_{i} \int_{r_{l-1}}^{r_{l}} r S_{imp}(r) \bar{\Theta}_{im} \, dr \right]/\left[ S_{imp}(\alpha_{imp}) \right].\]

Now using Eq.(2.49), we get

\[c_{mp}(t) = \left[ \bar{\Theta}_{mp}^{\psi} \right]/\left[ S_{imp}(\alpha_{imp}) \right].\]

Using Eq.(2.54) in Eq.(2.31), we get

\[\Theta_{j}(r,\theta,t) = \sum_{p=1}^{\infty} \Psi_{2j} S_{0j,p}(r) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{3j} S_{imp}(r) \cos(m\theta) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{4j} S_{imp}(r) \sin(m\theta),\] \hspace{1cm} (2.55)

where

\[\Psi_{2} = \left[ \bar{\Theta}_{mp}^{\psi} \right]/\left[ S_{0j,p}(\alpha_{0j,p}) \right], \quad \Psi_{3} = \left[ \bar{\Theta}_{mp}^{\psi} \right]/\left[ S_{imp}(\alpha_{imp}) \right], \quad \Psi_{4} = \left[ \bar{\Theta}_{mp}^{\psi} \right]/\left[ S_{imp}(\alpha_{imp}) \right].\]

Applying inverse Fourier Sine transform on the above Eq.(2.55), yields

\[\Theta_{j}(r,z,\theta,t) = \sum_{n=1}^{\infty} \left\{ \sum_{p=1}^{\infty} \Psi_{2j} S_{0j,p}(r) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{3j} S_{imp}(r) \cos(m\theta) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{4j} S_{imp}(r) \sin(m\theta) \right\} \sin(n\pi z / h).\] \hspace{1cm} (2.56)

Heat conductivity is taken as [1]:

\[\lambda_{j}(T_{j}) = \lambda_{i0} \exp(\Theta_{j} T_{j}), \quad \Theta_{j} \leq \theta .\] \hspace{1cm} (2.57)

Here \(\lambda_{i0}\) is the dimensionless reference value of thermal conductivity defined by,

\[\lambda_{i0}^{*} = \frac{\lambda_{i0}}{\lambda_{(i-1)0}} .\]

Substituting Eq.(2.57) in Eq.(2.17), yields

\[\Theta_{j} = (\lambda_{i0} / \Theta_{j})\{\exp(\Theta_{j} T_{j}) - 1\}.\] \hspace{1cm} (2.58)

Using Eq.(2.58) in Eq.(2.56), yields

\[T_{j}(r,z,\theta,t) = (l / \Theta_{j}) \log_{e}[g(r,z,\theta,t) + l] .\] \hspace{1cm} (2.59)
where
\[
g(r,z,\theta,t) = \sum_{n=1}^{\infty} \left( \frac{\Theta_1}{\lambda_1} \right)^n \left( \sum_{p=1}^{\infty} \Psi_{2S_{0P}}(r) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{3S_{imp}}(r) \cos(m\theta) + \right. \\
+ \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{4S_{imp}}(r) \sin(m\theta) \left. \right) \sin(n\pi z / h).
\]

We use the following logarithmic expansion
\[
\log \left[ g(r,z,\theta,t) + 1 \right] = [ g(r,z,\theta,t) + 1 / 2 ] \left[ g(r,z,\theta,t) \right]^2 + (1 / 3) \left[ g(r,z,\theta,t) \right]^3 + \ldots
\]

Ignoring terms with order greater than one, we get:
\[
\log \left( g(r,z,\theta,t) + 1 \right) = g(r,z,\theta,t).
\]

Hence Eq.(2.59) becomes
\[
T_i(r,z,\theta,t) = \sum_{n=1}^{\infty} \left( 1 / \lambda_1 \right)^n \left( \sum_{p=1}^{\infty} \Psi_{2S_{0P}}(r) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{3S_{imp}}(r) \cos(m\theta) + \right. \\
+ \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_{4S_{imp}}(r) \sin(m\theta) \left. \right) \sin(n\pi z / h).
\]

3. Thermoelastic analysis

In the cylindrical coordinate system, the boundary conditions for thin disc with support at both ends are [35]:
\[
\nabla^2 \nabla^2 w^{(i)} = \frac{-1}{(1-\nu_1)D_t} \nabla^2 M^{(i)}_T
\]

where
\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \quad D_t = \frac{E_t h^3}{12(1-\nu_t^2)}.
\]

Subject to conditions
\[
w^{(i)} = \frac{\partial w^{(i)}}{\partial t} = 0, \quad \text{at} \quad t=0.
\]

The problem is restricted under thermal load by an elastic reaction along the boundaries \( r=r_0, \quad r=r_k \), [16, 22]
\[
\frac{\partial w^{(i)}}{\partial r} + w^{(i)} = 0, \quad \text{at} \quad r=r_0,
\]
\[
\frac{\partial w^{(k)}}{\partial r} + w^{(k)} = 0, \quad \text{at} \quad r=r_k,
\]
\[
w^{(i)} = w^{(i-1)}, \quad \text{at} \quad r=r_{i-1},
\]
\[
\frac{\partial w^{(i)}}{\partial r} = \frac{\partial w^{(i-1)}}{\partial r} \quad \text{at} \quad r=r_{i-1}.
\] (3.4)

It is assumed that the constants of proportionality specified as per Hooke’s law are one. Resultant and shearing forces, moments are [35]:
\[
N^{(i)}_{rr} = N^{(i)}_{r0} = N^{(i)}_{r0} = 0,
\] (3.5)
\[
Q^{(i)}_{rr} = -D_i \frac{\partial}{\partial r} \left( \nabla^2 w^{(i)} \right) - \frac{1}{1-\nu_i} \frac{\partial M^{(i)}_{T}}{\partial r},
\] (3.6)
\[
Q^{(i)}_{r0} = -D_i \frac{\partial}{\partial \theta} \left( \nabla^2 w^{(i)} \right) - \frac{1}{1-\nu_i} \left( \frac{1}{r} \frac{\partial M^{(i)}_{T}}{\partial \theta} \right),
\]
\[
M^{(i)}_{rr} = -D_i \left[ \frac{\partial^2 w^{(i)}}{\partial r^2} + \frac{\nu_i}{r} \frac{\partial w^{(i)}}{\partial r} \right] - \frac{1}{1-\nu_i} M^{(i)}_{T},
\] (3.7)
\[
M^{(i)}_{r0} = -D_i \left[ \nu_i \frac{\partial^2 w^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial w^{(i)}}{\partial r} \right] - \frac{1}{1-\nu_i} M^{(i)}_{T}.
\]

Here \( M^{(i)}_{rr} \) satisfies the condition
\[
M^{(i)}_{rr} \bigg|_{r=r_0} = 0, \quad 0 < \theta < 1. \] (3.8)

Stress components are
\[
\sigma^{(i)}_{rr} = \frac{1}{h} N^{(i)}_{rr} + \frac{12z}{h^3} M^{(i)}_{rr} + \frac{1}{1-\nu_i} \left( \frac{1}{h} N^{(i)}_{T} + \frac{12z}{h^3} M^{(i)}_{T} - \alpha_i(T_i) E_i T_i \right),
\] (3.9)
\[
\sigma^{(i)}_{r0} = \frac{1}{h} N^{(i)}_{r0} + \frac{12z}{h^3} M^{(i)}_{r0} + \frac{1}{1-\nu_i} \left( \frac{1}{h} N^{(i)}_{T} + \frac{12z}{h^3} M^{(i)}_{T} - \alpha_i(T_i) E_i T_i \right),
\]
\[ M_{T}^{(i)} = E_i \int_0^h \alpha_i(T_i) T_i z \, dz, \quad N_{T}^{(i)} = E_i \int_0^h \alpha_i(T_i) T_i \, dz. \]  

(3.10)

Here \( \alpha_i(T_i) \) is the temperature dependent coefficient of linear thermal expansion assumed as:

\[ \alpha_i(T_i) = \alpha_{i\theta} \exp(\varpi_2 T_i), \quad \varpi_2 \geq 0. \]  

(3.11)

Here \( \alpha_{i\theta} \) is the dimensionless reference value of coefficient of linear thermal expansion defined by,

\[ \alpha_{i\theta}^* = \frac{\alpha_{i\theta}}{\alpha_{(i-1)\theta}}. \]

Using Eqs (2.61) and (3.11), in Eq.(3.10), we get \( M_{T}^{(i)} \) and \( N_{T}^{(i)} \) as:

\[ M_{T}^{(i)} = \left( \alpha_{i} E_i / \lambda_{i\theta} \right) \sum_{n=1}^{\infty} \Psi_5 \left\{ \sum_{p=1}^{\infty} \Psi_2 S_{i0p}(r) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_3 S_{i\text{imp}}(r) \cos(m\theta) + \right. \\
\left. + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_4 S_{i\text{imp}}(r) \sin(m\theta) \right\}, \]

(3.12)

\[ N_{T}^{(i)} = \left( \alpha_{i} E_i / \lambda_{i\theta} \right) \sum_{n=1}^{\infty} \Psi_6 \left\{ \sum_{p=1}^{\infty} \Psi_2 S_{i0p}(r) + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_3 S_{i\text{imp}}(r) \cos(m\theta) + \right. \\
\left. + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_4 S_{i\text{imp}}(r) \sin(m\theta) \right\}, \]

(3.13)

where

\[ \Psi_5 = \left( h^2 / 24n^2\pi^2 \right) \left[ (-3n\pi(8 + 3\varpi_2^2) \cos(n\pi) + \varpi_2 (3 + 6n^2\pi^2 - 3\cos(2n\pi)) + n^2\varpi_2^2 \cos(3n\pi)) \right], \]

\[ \Psi_6 = \left( h / 4n\pi \right) \left[ 24 + 12n\varpi_2^2 + 8\varpi_2^2 - 3(8 + 3\varpi_2^2) \cos(n\pi) + \varpi_2^2 \cos(3n\pi) \right]. \]

Using Eq.(3.12), the deflection \( w^{(i)} \) of the \( i^{th} \) layer from Eq.(3.1) is obtained as:

\[ w^{(i)} = \left\{ 12(1 + \nu)\alpha_i \Psi_5 / \lambda_{i\theta} h^3 \right\} \left[ \left( \Phi_j + \Phi_2 + \Phi_3 \right) / \left( \Phi_j + ((m^2 + r^2) / r^2) \right) \Phi_2 + \right. \\
\left. + \left( (m^2 + r^2) / r^2 \right) \Phi_3 \right] \times \sum_{n=1}^{\infty} \left\{ \Phi_j S_{i0p}(r) + \Phi_2 S_{i\text{imp}}(r) + \Phi_3 S_{i\text{imp}}(r) \right\}, \]

(3.14)

where

\[ \Phi_j = \sum_{p=1}^{\infty} \Psi_2, \quad \Phi_2 = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_3 \cos(m\theta), \quad \Phi_3 = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \Psi_4 \sin(m\theta). \]

Using Eqs (3.12) and (3.14), Eqs (3.7), (3.9) are solved with the help of Mathematica software.
4. Numerical analysis

For numerical analysis, a mathematical model is formulated for a 3-layered disc, with the inner, middle and outer layers composed of copper, zinc and aluminum, respectively [22]. Let ambient temperature $T_0 = 20$, $r_0 = 1$, $r_1 = 2$, $r_2 = 3$, $r_3 = 4$, inner layer $r_0 < r < r_1$, middle layer $r_1 < r < r_2$, outer layer $r_2 < r < r_3$.

Figures 2(a), 2(b), 2(c) shows temperature along $\theta, z, r$ respectively. Here temperature distribution follows a sinusoidal nature. Along axial direction, its magnitude suddenly increases and reaches to zero towards the end. Along radial direction, the temperature slowly increases and attains peak and reduces towards the inner-layer.

The following Figs (3-7) on left represent homogeneous case, while on right represent nonhomogeneous case. Figures 3(a), 3(b) show the dimensionless deflection along $\theta, r$ respectively. Due to the application of heat, the deflection is more in outer-layer as compared to other layers. Its magnitude is more in the temperature dependent case as compared to the temperature independent case.

Figures 4(a), 4(b), 5(a), 5(b) show the dimensionless resultant moments along $\theta, r$. The moment $M_{\theta\theta}$ is tensile in nature, while $M_{rr}$ is seen to be tensile and compressive in different regions.

Figures 6(a), 6(b), 6(c) show $\sigma_{\theta\theta}$, while Figs 7(a), 7(b), 7(c) show $\sigma_{rr}$ along $\theta, z, r$. $\sigma_{\theta\theta}$ is tensile along $\theta, z$, while radially its nature changes in both cases. Magnitude of $\sigma_{rr}$ gradually increases with increase in $\theta, z$, while along $r$ direction it becomes compressive and attains tensile nature towards the inner-layer.

Fig.3(a). Plot of deflection along $\theta$. 
Thermal stresses associated with a thermosensitive multilayered...
Fig. 5(b). Plot of $M_{rr}$ along $r$.

Fig. 6(a). Plot of $\sigma_{\theta\theta}$ along $\theta$.

Fig. 6(b). Plot of $\sigma_{\theta\theta}$ along $z$.

Fig. 6(c). Plot of $\sigma_{\theta\theta}$ along $r$. 
5. Validation of the results

This paper presents a mathematical model for a multilayered thin annular circular disc. The asymmetric heat conduction problem with time-dependent boundary conditions and heat source is solved using the finite integral transform approach. The temperature distribution, along with its corresponding deflection, resultant moments, and thermal stress distributions, has been derived. As a limiting case, if we consider homogeneous material properties, the results agree with [22].

6. Conclusion

Thermal behavior of a multiple layered annular circular disc due to instantaneous point heating is investigated in this paper by taking thermally sensitive material properties. The HCE is solved using
Kirchhoff’s method and finite integral transform method. In the temperature independent case, the radial stress suddenly changes to compressive nature as the heat passes from middle layer to inner layer, whereas it is tensile in nature for all regions in the temperature dependent case. The point heat source generates heat in the annular disc in the middle layer. This heat source causes the temperature rise in the middle layer, while heat propagates accordingly in the inner and outer layers. The annular disc experiences notable variations in thermoelastic properties due to the introduction of point heat source.

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Nomenclature

$C_i(T_i)$ – specific heat capacity of the $i$th layer
$D_i$ – bending rigidity of the $i$th layer
$E_i$ – Young’s modulus of the $i$th layer
$h$ – thickness of the disc
$h_{0i}, h_{Ki}$ – surface coefficients
$M_{rr}^{(i)}, M_{\theta\theta}^{(i)}$ – resultant moments
$M_T^{(i)}, N_T^{(i)}$ – thermally induced resultant moments of the $i$th layer
$N_{rr}^{(i)}, N_{\theta\theta}^{(i)}$ – resultant forces
$Q(r, \theta, z, t)$ – internal heat generation
$Q_{rr}^{(i)}, Q_{\theta\theta}^{(i)}$ – shearing forces
$T_i$ – temperature of the $i$th layer
$T_0$ – ambient temperature
$w^{(i)}$ – deflection of the $i$th layer
$\alpha_i(T_i)$ – temperature dependent coefficient of linear thermal expansion
$\lambda_i(T_i)$ – thermal conductivity of the $i$th layer
$\nu_i$ – Poisson’s ratio of the $i$th layer
$\rho_i$ – density of the $i$th layer
$\sigma_{rr}, \sigma_{\theta\theta}$ – components of stress functions

References

Thermal stresses associated with a thermosensitive multilayered...

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