IJAME

Int. J. of Applied Mechanics and Engineering, 2024, vol.29, No.X, pp.16-25
DOI: 10.59441/ijame/187049

ACCURATE COMPUTATIONAL APPROACH FOR SINGULARLY PERTURBED BURGER-HUXLEY EQUATIONS

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The main purpose of this work is to present an accurate computational approach for solving the singularly perturbed Burger-Huxley equations. The quasilinearization technique linearizes the nonlinear term of the differential equation. The finite difference approximation is formulated to approximate the derivatives in the differential equations and then accelerate its rate of convergence to improve the accuracy of the solution. Numerical experiments were conducted to sustain the theoretical results and to show that the presented method produces a more correct solution than some surviving methods in the literature.

Key words: singularly perturbed, nonlinear term, Burger-Huxley equations, accurate solution.

1. Introduction

In a real-life phenomenon, mathematical modeled problems typically result in functional equations, like ordinary or partial differential equations (PDEs), integral and integrodifferential equations, and stochastic equations. In this day and age, PDEs have converted into an appropriate device for telling the natural phenomena of science, engineering models, and the marvels arising in mathematical physics. In solving different types of singularly perturbed parabolic PDEs, most numerical methods face divergence and instability due to the existence of the nonlinear term, and perturbation parameter, [1-3].

Numerical methods are techniques that are used to provide approximate solutions for mathematical modeled problems. We need approximations since we either cannot solve analytically or the exact method is intractable. Hence, researchers are working on the solution methodologies of singularly perturbed Burger-Huxley equations, [4,5]. Thus, researchers tried to develop different methods to solve these equations that served as a ground to get prior knowledge and the limitations of the existing methods. Basically, from the review of existing literature, we observe a good thing: the procedure to consider initial guesses to linearize the non-linear term, and higher-order numerical methods are preferred as they provide more accurate solutions with low computational cost so that they are efficient methods, [6-9]. Moreover, it is a recent and active research area in engineering and science fields that arises in various physical models including heat supply in the human head, the process of oxygen diffusion, the growth of some tumor types, and so on, [10-17].

However, due to the occurrence of nonlinear terms and small positive parameters in the differential equation, it is problematic to get a more accurate solution as much as required unless using a large number of mesh points. These show that the existing methods are computationally costly and inefficient. Thus, it forces us to think of alternate numerical procedures to solve the problem under consideration. Therefore, the objective of this paper is to provide an accurate computational method for solving singularly perturbed Burger-Huxley equations.

Thus, we construct an accurate computational approach for solving the singularly perturbed Burger-Huxley equations:

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$$\begin{cases} -\varepsilon\dfrac{\partial^2 y}{\partial x^2}+\alpha y\dfrac{\partial y}{\partial x}+\dfrac{\partial y}{\partial t}-\beta(1-y)(y-\gamma)y=0, \quad \forall(x,t)\in D=(0,1)\times(0,T], \\ y(x,0)=y_0(x), \quad x\in[0,1], \\ y(0,t)=q_0(t), \quad t\in(0,T], \\ y(1,t)=q_1(t), \quad t\in(0,T], \end{cases} \tag{1.1}$$

where ε, $(0<\varepsilon<1)$, is a perturbation parameter with T a positive number, and the given constants satisfy the conditions $\alpha\geq 1$, $\beta\geq 0$, $\gamma\in(0,1)$. Further, the concerned researcher can find a real-time application of the modeled problem of Eq.(1.1) and its analytical properties such as the existence, stability, and boundness of the solutions in the literature [1-6]

2. Formulation of the numerical scheme

Re-writing the defined problem in Eq.(1.1) as:

$$\begin{cases} \dfrac{\partial y}{\partial t}-\varepsilon\dfrac{\partial^2 y}{\partial x^2}=F(x,t,y,\dfrac{\partial y}{\partial x}), \quad \forall(x,t)\in D, \\ y(x,0)=y_0(x), \quad x\in[0,1], \\ y(0,t)=q_0(t), \quad t\in(0,T], \\ y(1,t)=q_1(t), \quad t\in(0,T], \end{cases} \tag{2.1}$$

for the nonlinear terms are denoted by $F\left(x,t,y,\dfrac{\partial y}{\partial x}\right)=\beta(1-y)(y-\gamma)y-\alpha y\dfrac{\partial y}{\partial x}$.

When the constants in the nonlinear term are $\alpha=\beta=0$, then Eq.(2) becomes:

$$\begin{cases} \dfrac{\partial y}{\partial t}-\varepsilon\dfrac{\partial^2 y}{\partial x^2}=0 \quad \forall(x,t)\in D, \\ y(x,0)=y_0(x), \quad x\in[0,1], \\ y(0,t)=q_0(t), \quad t\in(0,T], \\ y(1,t)=q_1(t), \quad t\in(0,T]. \end{cases} \tag{2.2}$$

From the works in [2, 3, 4], for the parameter, $\varepsilon=1$, the solution for Eq.(2.2), is given by:

$$y(x,t)=y_0(x)\exp(-\pi^2 t). \tag{2.3}$$

Following the works by Kabeto, and Duressa [2-4], to linearize Eq.(2.1), by applying the quasilinearization technique on the nonlinear term, for the reasonable initial guess of the form of Eq.(2.3) written as:

$$y^{(0)}(x,t)=y_0(x)\exp(-\pi^2 t), \tag{2.4}$$

Accordingly, the nonlinear term, initially linearized as:

$$F\left(x,t,y,\frac{\partial y}{\partial x}\right) \cong F\left(x,t,y^{(0)},\frac{\partial y^{(0)}}{\partial x}\right) + \left(y^{(1)} - y^{(0)}\right)\frac{\partial F}{\partial y}\bigg|_{y^{(0)}} + \left(\frac{\partial y^{(1)}}{\partial x} - \frac{\partial y^{(0)}}{\partial x}\right)\frac{\partial F}{\partial\left(\frac{\partial y}{\partial x}\right)}\Bigg|_{\frac{\partial y^{(0)}}{\partial x}} + \ldots \quad (2.5)$$

Considering the expanded terms up to the first order from Eq.(2.5) into Eq.(2.1) and persuading for iteration number i, yields:

$$\frac{\partial y^{(i+1)}}{\partial t} - \varepsilon\frac{\partial^2 y^{(i+1)}}{\partial x^2} + a^{(i)}(x,t)\frac{\partial y^{(i+1)}}{\partial x} + b^{(i)}(x,t)y^{(i+1)} = f^{(i)}(x,t), \quad (2.6)$$

where $$a^{(i)}(x,t) = -\frac{\partial F}{\partial\left(\frac{\partial y}{\partial x}\right)}\Bigg|_{\left(x,t,y^{(i)},\frac{\partial y^{(i)}}{\partial x}\right)}, \quad b^{(i)}(x,t) = -\frac{\partial F}{\partial y}\bigg|_{\left(x,t,y^{(i)},\frac{\partial y^{(i)}}{\partial x}\right)},$$

$$f^{(i)}(x,t) = F\left(x,t,y^{(i)},\frac{\partial y^{(i)}}{\partial x}\right) - y^{(i)}\frac{\partial F}{\partial y}\bigg|_{\left(x,t,y^{(i)},\frac{\partial y^{(i)}}{\partial x}\right)} - \frac{\partial y^{(i)}}{\partial x}\frac{\partial F}{\partial\left(\frac{\partial y}{\partial x}\right)}\Bigg|_{\left(x,t,y^{(i)},\frac{\partial y^{(i)}}{\partial x}\right)}.$$

Due to the layer behaviors, in the theory of solving singularly perturbed problems, the standard numerical methods are challenges for solving these problems. However, the linearized form of the singularly perturbed Burger-Huxley equation in Eq.(2.6) behaves weak layer behavior because of the coefficient of the convection term satisfy $a^{(i)}(x,t) \geq 0$. Thus, the standard higher-order numerical method works for singularly perturbed problems caused by weak layers.

Now, let M and N be positive integers, and use a rectangular grid D_h^k whose nodes are $\left(x_m,t_n\right)$ defined by:

$$\begin{cases} t_n = nk, \ \ k = \dfrac{T}{N}, \ \ n = 0,1,2,\ldots,N, \\ x_m = mh, \ \ h = \dfrac{1}{M}, \ \ m = 0,1,2,\ldots,M. \end{cases} \quad (2.7)$$

Adopt the approximate solution $y_m^n \approx y(x_m,t_n)$ at an arbitrary point (x_m,t_n), and the differential equation in Eq.(2.6) is satisfied at the $\left(m, n+\frac{1}{2}\right)^{th}$ level. Then, at the first linearizing iteration $i = 0$, Eq.(2.6) can be written as:

$$\frac{\partial y_m^{n+\frac{1}{2}}}{\partial t} - \varepsilon\frac{\partial^2 y_m^{n+\frac{1}{2}}}{\partial x^2} + a_m^{n+\frac{1}{2}}\frac{\partial y_m^{n+\frac{1}{2}}}{\partial x} + b_m^{n+\frac{1}{2}}y_m^{n+\frac{1}{2}} = f_m^{n+\frac{1}{2}}. \quad (2.8)$$

Now, by Taylor series expansion, derivatives in $t-$direction, becomes:

$$y_m^{n+1} = y_m^{n+\frac{1}{2}} + \frac{k}{2}\frac{\partial y_m^{n+\frac{1}{2}}}{\partial t} + \frac{k^2}{8}\frac{\partial^2 y_m^{n+\frac{1}{2}}}{\partial t^2} + \frac{k^3}{48}\frac{\partial^3 y_m^{n+\frac{1}{2}}}{\partial t^3} + O\left(k^4\right), \tag{2.9}$$

$$y_m^{n} = y_m^{n+\frac{1}{2}} - \frac{k}{2}\frac{\partial y_m^{n+\frac{1}{2}}}{\partial t} + \frac{k^2}{8}\frac{\partial^2 y_m^{n+\frac{1}{2}}}{\partial t^2} - \frac{k^3}{48}\frac{\partial^3 y_m^{n+\frac{1}{2}}}{\partial t^3} + O\left(k^4\right). \tag{2.10}$$

Subtracting Eq.(2.10) from Eq.(2.9), yields,

$$\frac{\partial y_m^{n+\frac{1}{2}}}{\partial t} = \frac{y_m^{n+1} - y_m^n}{k} + \tau_1 \tag{2.11}$$

where the truncation term $\tau_1 = -\frac{k^2}{24}\frac{\partial^3 y_m^{n+\frac{1}{2}}}{\partial t^3}$.

Besides, the remaining terms of Eq.(2.8), on average, are written as:

$$-\varepsilon\frac{\partial^2 y_m^{n+\frac{1}{2}}}{\partial x^2} + a_m^{n+\frac{1}{2}}\frac{\partial y_m^{n+\frac{1}{2}}}{\partial x} + b_m^{n+\frac{1}{2}} y_m^{n+\frac{1}{2}} - f_m^{n+\frac{1}{2}} = \frac{L_x^N y_m^{n+1} + L_x^N y_m^n}{2} \quad , \tag{2.12}$$

$$L_x^N y_m^{n+1} = -\varepsilon\frac{y_{m+1}^{n+1} - 2y_m^{n+1} + y_{m-1}^{n+1}}{h^2} + a_m^{n+1}\frac{y_{m+1}^{n+1} - y_{m-1}^{n+1}}{2h} + b_m^{n+1} y_m^{n+1} - f_m^{n+1},$$

for

$$L_x^N y_m^{n} = -\varepsilon\frac{y_{m+1}^{n} - 2y_m^{n} + y_{m-1}^{n}}{h^2} + a_m^{n}\frac{y_{m+1}^{n} - y_{m-1}^{n}}{2h} + b_m^{n} y_m^{n} - f_m^{n} + \tau_2,$$

$$\tau_2 = h^2\left(\frac{\varepsilon}{12}\frac{\partial^4 y_m^{n+1}}{\partial x^4} - \frac{a_m^{n+1}}{6}\frac{\partial^4 y_m^{n+1}}{\partial x^4} + \frac{\varepsilon}{12}\frac{\partial^3 y_m^{n}}{\partial x^3} - \frac{a_m^{n}}{6}\frac{\partial^4 y_m^{n}}{\partial x^4}\right).$$

Substituting Eqs. (2.11) and (2.12) into Eq.(2.8) gives:

$$\begin{aligned} &2y_m^{n+1} - \frac{\varepsilon k}{h^2}\left(y_{m+1}^{n+1} - 2y_m^{n+1} + y_{m-1}^{n+1}\right) + \frac{ka_m^{n+1}}{2h}\left(y_{m+1}^{n+1} - y_{m-1}^{n+1}\right) + kb_m^{n+1} y_m^{n+1} = \\ &= 2y_m^n + \frac{\varepsilon k}{h^2}\left(y_{m+1}^{n} - 2y_m^{n} + y_{m-1}^{n}\right) - \frac{ka_m^{n}}{2h}\left(y_{m+1}^{n} - y_{m-1}^{n}\right) - kb_m^n y_m^n + k\left(f_m^{n+1} + f_m^n\right) + \tau_3 \end{aligned} \tag{2.13}$$

where $\tau_3 = 2\left(\tau_1 + \tau_2\right)$.

The three-term recurrence relation in the spatial direction is written as:

$$E_m^{n+1} y_{m-1}^{n+1} + F_m^{n+1} y_m^{n+1} + G_m^{n+1} y_{m+1}^{n+1} = H_m^{n+1}, \tag{2.14}$$

where $$E_m^{n+1} = -\frac{\varepsilon k}{h^2} - \frac{k a_m^{n+1}}{2h}, \quad F_m^{n+1} = 2 + \frac{2\varepsilon k}{h^2} k b_m^{n+1}, \quad G_m^{n+1} = -\frac{\varepsilon k}{h^2} + \frac{k a_m^{n+1}}{2h},$$

$$H_m^{n+1} = k\left(f_m^{n+1} + f_m^n\right) + 2y_m^n + \frac{\varepsilon k}{h^2}\left(y_{m+1}^n - 2y_m^n + y_{m-1}^n\right) - \frac{k a_m^n}{2h}\left(y_{m+1}^n - y_{m-1}^n\right) - k b_m^n y_m^n .$$

The coefficients E_m^{n+1}, F_m^{n+1}, G_m^{n+1} are assumed to be satisfying the following conditions:

$$\left|E_m^{n+1}\right| > 0, \ \left|F_m^{n+1}\right| > 0, \ \left|G_m^{n+1}\right| > 0 \quad \text{and} \quad \left|F_m^{n+1}\right| > \left|E_m^{n+1}\right| + \left|G_m^{n+1}\right| .$$

Diagonally dominant in the spatial direction at each $(n+1)^{th}$ time level is ensured by this condition verified as:

$$2 + \frac{2\varepsilon k}{h^2} + k b_m^{n+1} \ \geq \ \frac{\varepsilon k}{h^2} + \frac{k a_m^{n+1}}{2h} + \frac{\varepsilon k}{h^2} - \frac{k a_m^{n+1}}{2h} .$$

Therefore, solving Eq.(2.15) and the given boundary conditions by the Thomas algorithm is stable.

3. Convergence analysis

To investigate the consistency of the method, consider the truncation errors given in Eqs. (2.11), (2.12) and (2.13) written as:

$$T_m^{n+1} = \tau_3 = 2\left(\tau_1 + \tau_2\right), \tag{3.1}$$

where $$\tau_1 = -\frac{k^2}{24}\frac{\partial^3 y_m^{n+\frac{1}{2}}}{\partial t^3} \quad \text{and} \quad \tau_2 = h^2\left(\frac{\varepsilon}{12}\frac{\partial^4 y_m^{n+1}}{\partial x^4} - \frac{a_m^{n+1}}{6}\frac{\partial^4 y_m^{n+1}}{\partial x^4} + \frac{\varepsilon}{12}\frac{\partial^3 y_m^n}{\partial x^3} - \frac{a_m^n}{6}\frac{\partial^4 y_m^n}{\partial x^4}\right).$$

Hence, Eq.(3.1) vanishes as $k \to 0$ and $h \to 0$. Therefore, the scheme developed in Eq.(2.14), is a consistent and stable finite difference method that implies is convergent by Lax's equivalence theorem, [11, 12]. Thus, the truncation error in Eq.(2.13) becomes:

$$T(h,k) \equiv O\left(h^2 + k^2\right). \tag{3.2}$$

Further, the truncation error norm is:

$$\left|T(h,k)\right| = \left|y(x_m, t_{n+1}) - y_m^{n+1}\right| \ \leq \ C_1 k^2 + C_2 h^2 \tag{3.3}$$

where the constants obtained from the maximum norm are:

$$C_1 = \frac{1}{24}\left\|\frac{\partial^3 y_m^{n+\frac{1}{2}}}{\partial t^3}\right\|_\infty , \quad \text{and} \quad C_2 = \left\|\frac{\varepsilon}{12}\frac{\partial^4 y_m^{n+1}}{\partial x^4} - \frac{a_m^{n+1}}{6}\frac{\partial^4 y_m^{n+1}}{\partial x^4} + \frac{\varepsilon}{12}\frac{\partial^3 y_m^n}{\partial x^3} - \frac{a_m^n}{6}\frac{\partial^4 y_m^n}{\partial x^4}\right\|_\infty .$$

To increase the efficiency of the formulated scheme to produce a more accurate solution, accelerate the rate of convergence by applying the Richardson extrapolation techniques. Henceforth, assume that $y(h,k)$ denote the approximate value of $y(x_m,t_n)$ with the mesh length of h and k. Another approximate solution $y\left(\frac{h}{2},\frac{k}{2}\right)$ also denotes the value of $y(x_m,t_n)$ obtained by using the same method with step length $\frac{h}{2}$ and $\frac{k}{2}$, then by the double mesh principle, we have:

$$\begin{cases} y(x_m,t_n)-y(h,k)\equiv C\left(h^2+k^2\right)+O\left(h^4+k^4\right), \\ y(x_m,t_n)-y\left(\frac{h}{2},\frac{k}{2}\right)\equiv C\left(\frac{h^2}{4}+\frac{k^2}{4}\right)+O\left(h^4+k^4\right), \end{cases} \tag{3.4}$$

for C is a constant free from the mesh parameters h and k.
To eliminate C in Eq.(3.4), multiply the second equation by 4, and then subtracting the first from the second one provides:

$$y(x_m,t_n)-y^{ext}_{(h,k)}\equiv O\left(h^4+k^4\right), \tag{3.5}$$

where $$y^{ext}_{(h,k)}=\frac{4y\left(\frac{h}{2},\frac{k}{2}\right)-y(h,k)}{3}.$$

By this approximation, the estimated truncation error is:

$$\left|y(x_m,t_n)-y^{ext}_{(h,k)}\right|\leq C\left(h^4+k^4\right). \tag{3.6}$$

4. Numerical illustrations

In this section, we have chosen two model examples of singularly perturbed Burger-Huxley equations, since they have been widely discussed in the literature to validate our theoretical results. The maximum absolute errors are calculated by the double mesh principle [2-4], which is given by:

$$\left(E_M^N\right)^{ext}=\max_{\forall(x_m,t_n)\in\bar{D}}\left|\left(y_m^n\right)^{ext}-\left(y_{2m}^{2n}\right)^{ext}\right|,$$

where y_m^n y_{2m}^{2n} and are approximate solutions evaluated on $\bar{D}_M^N$ $\bar{D}_{2M}^{2N}$ and respectively. The corresponding rate of convergences is:

$$\left(R_M^N\right)^{ext}=\frac{\log\left[\left(E_M^N\right)^{etx}\right]-\log\left[\left(E_{2M}^{2N}\right)^{ext}\right]}{\log(2)}.$$

Example 1: Consider the following problem:

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 y(x,t)}{\partial x^2} + y(x,t)\dfrac{\partial y(x,t)}{\partial x} + [1-y(x,t)][y(x,t)-0.5]y(x,t) = 0, \quad (x,t)\in(0,1)\times(0,1], \\ y(x,0) = x(1-x^2), \quad 0<x<1, y(0,t) = y(1,t) = 0, \quad t\in[0,1]. \end{cases}$$

Example 2: Consider the following problem:

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 y(x,t)}{\partial x^2} + y(x,t)\dfrac{\partial y(x,t)}{\partial x} = 0, \quad (x,t)\in(0,1)\times(0,1], \\ y(x,0) = x(1-x^2), \quad 0<x<1, \qquad y(0,t) = y(1,t) = 0, \quad t\in[0,1]. \end{cases}$$

Table 1. Computed maximum absolute errors and rate of convergence for Example 1.

$\varepsilon \downarrow M = N \rightarrow$	8	16	32	64	128
2^{-8}	1.2278e-04	2.2830e-05	5.8021e-06	1.4939e-06	3.7377e-07
	2.4271	1.9763	1.9575	1.9989	
2^{-10}	1.2862e-04	2.0187e-05	6.9973e-06	1.5116e-06	3.6589e-07
	2.6716	1.5286	2.2107	2.0466	
2^{-12}	1.3259e-04	9.5377e-06	5.9545e-06	1.8183e-06	3.8349e-07
	3.7972	0.6797	1.7114	2.2453	
2^{-18}	1.3435e-04	9.5457e-06	6.3330e-07	6.8551e-08	1.1037e-08
	3.8150	3.9139	3.2076	2.6348	

Table 2. Contrast of maximum absolute errors for Example 1.

$\varepsilon \downarrow M / N \rightarrow$	32/20	64/40	128/80	256/160
Present Method				
2^{-10}	5.2656e-06	1.4249e-06	3.7914e-07	4.1826e-08
2^{-12}	4.7092e-06	1.7519e-06	3.7869e-07	8.8200e-08
2^{-14}	4.3549e-06	1.4876e-06	4.5581e-07	9.6088e-08
2^{-16}	4.3616e-06	2.8334e-06	3.8419e-07	1.1468e-07
Result in [5]				
2^{-10}	2.5551e-04	6.9513e-05	1.8120e-05	4.4961e-06
2^{-12}	2.7297e-04	7.2989e-05	1.8782e-05	4.5990e-06
2^{-14}	2.7342e-04	7.3024e-05	1.8789e-05	4.6006e-06
2^{-16}	2.7377e-04	7.3049e-05	1.8794e-05	4.6016e-06
Result in [2]				
2^{-10}	2.9393e-04	7.9356e-05	2.0013e-05	5.0159e-06
2^{-14}	2.9695e-04	7.9882e-05	2.0146e-05	5.0535e-06
2^{-14}	2.9770e-04	8.0013e-05	2.0180e-05	5.0629e-06
2^{-16}	2.9789e-04	8.0046e-05	2.0188e-05	5.0653e-06

Table 3. Contrast of maximum absolute errors for Example 2.

$\varepsilon \downarrow M / N \rightarrow$	*32/20*	*64/40*	*128/80*	*256/160*
Present Method				
2^{-10}	*8.2942e-06*	*1.6285e-06*	*3.8589e-07*	*4.1945e-08*
2^{-12}	*1.0630e-05*	*2.5089e-06*	*4.2530e-07*	*9.1141e-08*
2^{-14}	*4.0174e-06*	*3.0865e-06*	*6.4493e-07*	*1.0713e-07*
2^{-16}	*4.0255e-06*	*2.6091e-06*	*7.8877e-07*	*1.6206e-07*
Result in [5]				
2^{-10}	*2.7265e-04*	*7.1303e-05*	*1.7154e-05*	*4.2520e-06*
2^{-12}	*2.7366e-04*	*7.2313e-05*	*1.8164e-05*	*4.3530e-06*
2^{-14}	*2.7381e-04*	*7.2463e-05*	*1.8371e-05*	*4.4612e-06*
2^{-16}	*2.7384e-04*	*7.2530e-05*	*1.8382e-05*	*4.4819e-06*
Result in [2]				
2^{-10}	*2.5222e-04*	*6.7547e-05*	*1.7077e-05*	*4.2892e-06*
2^{-12}	*2.5527e-04*	*6.8045e-05*	*1.7231e-05*	*4.3246e-06*
2^{-14}	*2.5603e-04*	*6.8169e-05*	*1.7270e-05*	*4.3334e-06*
2^{-16}	*2.5622e-04*	*6.8201e-05*	*1.7280e-05*	*4.3356e-06*

Table 1 shows the computed maximum absolute errors and the rate of convergence to validity the errors are monotonically decreasing behavior with an increasing number of intervals which confirms the convergence of the method. Tables 2 and 3, provide the comparisons of the obtained maximum absolute errors with the existing methods in the literature and to confirmation the superiority of the presented method. Further, these tables show the accurate computational approach is an efficient method for solving the considered problem that confirms the theoretical results. Thus, the method provides a more accurate solution than existing methods. Figure 1 authenticates the physical behavior of the solutions.

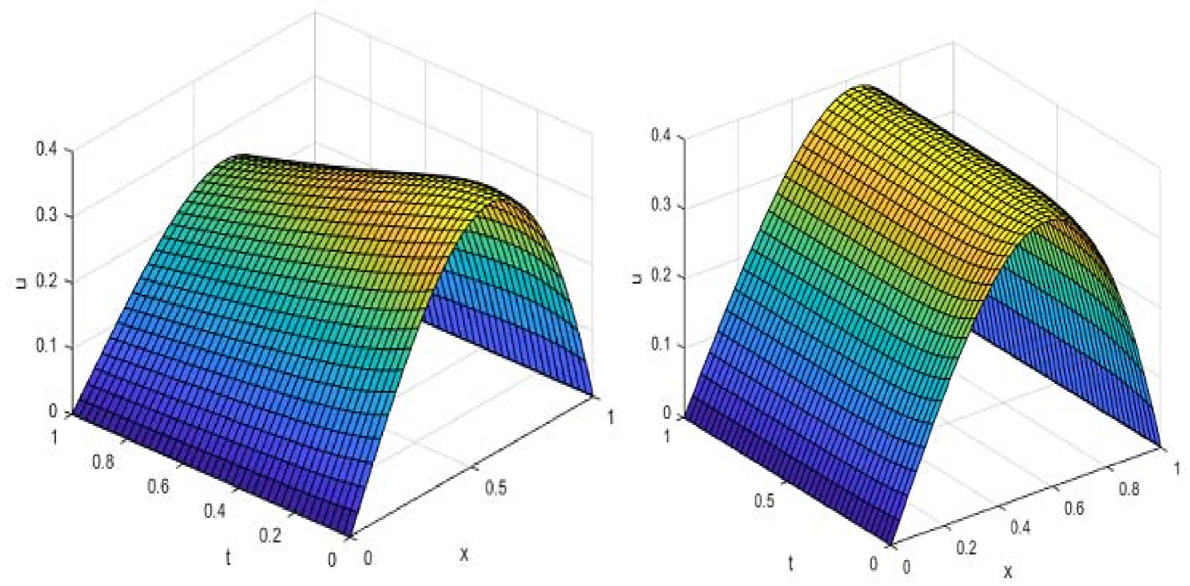

Fig.1. Solution profiles for the two Examples respectively, when $M = 64,\ N = 40,\ \varepsilon = 2^{-18}$.

5. Conclusion

In this paper, an accurate computational approach is offered for solving the singularly perturbed Burger-Huxley equations. The finite difference approximation methodology procedure is constructed and then accelerated to higher-order convergent to improve the accuracy of the solution. The efficiency of the method is illustrated by numerical results, which provide an accurate solution. Consequently, the computational approach produces an accurate solution for solving the singularly perturbed Burger-Huxley equations.

Acknowledgments

This work is supported by Jimma University, Ethiopia.

Nomenclature

D – solution domain.
h – mesh length in x – direction.
k – mesh length in t – direction.
M – the number of mesh intervals in the spatial direction.
N – the number of mesh intervals in the temporal direction.
O – order of convergence.
PDEs – partial differential equations.
ε – perturbation parameter.

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Received: January 9, 2024
Revised: April 11, 2024