ONE-DIMENSIONAL THERMAL SHOCK PROBLEM FOR A SEMI-INFINITE HYGROTHERMOELASTIC ROD

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The present research article deals with the study of a boundary value problem of a one-dimensional semi-infinite hygro-thermoelastic rod of length $l$. The deformation of the rod is under consideration when the left boundary of the hygro-thermoelastic rod is subjected to a sudden heat source. The solutions of the considered variables are decomposed in terms of normal modes. Analytical expressions of displacement, moisture concentration, temperature field, and stresses are obtained and presented graphically for different periods. By studying the one-dimensional thermal shock problem for a semi-infinite hygrothermoelastic rod, the authors aim to gain insights into the fundamental behavior of materials subjected to rapid temperature changes and moisture effects.

Key words: one-dimensional, heat source, temperature distribution, hygrothermoelastic rod, moisture concentration.

1. Introduction

The theory of dynamic thermoelasticity finds applications in different fields of science has been explored by researchers. The classical coupled theory of thermoelasticity was developed which is parabolic in nature and predicts infinite velocity of heat propagation thereby making it impossible to accept. To overcome this paradox, different theories of generalized thermoelasticity which involve a hyperbolic heat equation have been developed in the past few decades. This type of equation admits a finite speed of thermal signals. Chandrasekharaiah and Srinath [1] considered the linear theory of thermoelasticity without energy dissipation to discuss the effect of continuous point heat source in homogeneous and isotropic thermoelastic media. Sharma et al. [2] investigated a homogeneous isotropic thermoelastic half-space under the effect of a mechanical and thermal source. Baksi et al. [3] investigated a three-dimensional problem in a rotating magneto-thermoelastic medium with thermal relaxation. Mallik and Kanoria [4] discussed periodically varying heat source in a functionally graded isotropic thermoelastic medium. The problem of a moving heat source in a magneto-thermoelastic strip was discussed by He and Cao [5]. A problem of heat sources in a semi-infinite thermoelastic cylinder was discussed by Tripathi et al. [6]. The dynamic response of a semi-infinite thermoelastic plate by the finite element method was presented by Xia et al. [7]. Ailawalia and Budhiraja [8] studied deformation in a thermo-microstretch elastic medium underlying a non-viscous fluid layer subjected to an internal heat source. Abbas [9] applied the eigen-value approach in a fractional-order thermoelastic medium. Ailawalia and Sachdeva [10] presented the effect of an internal heat source in a thermoelastic solid with micro-temperatures. Sarkar and Mondal [11] inspected thermoelastic interactions in a slim strip under a dual-phase-lag model due to a moving heat source. They also discussed the transient responses in a two-temperature generalized thermoelastic infinite medium due to a time-dependent heat source [12]. Mondal et

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The interaction between heat, moisture, and deformation is considered an important issue in engineering science. Materials get damaged when moisture and temperature interact with mechanical stresses. These results suggest that more study is required for the interaction of moisture and temperature. In solids, atoms move randomly, and the average spacing between them remains constant as long as they vibrate about their equilibrium positions in a symmetric manner. Motion of particles from one part to another also occurs which is known as diffusion. Further exchange of kinetic energy occurs between adjacent particles in the form of heat which travels from high to low temperature. But it follows from the second law of thermodynamics that the exchange of heat can be reversible if heat flow occurs at an infinitely small rate in the presence of a temperature gradient. The principle of irreversible thermodynamics governs physiochemical processes such as heat conduction, diffusion, etc. Sih et al. [17] and Weitsman [18] developed coupled equations for a hygrothermoelastic medium keeping in view principles of irreversible thermodynamics and continuum mechanics. Basi et al. [19] analyzed anisotropic inhomogeneous and laminated plates under hygrothermoelastic effects. Fluid-saturated porous media subjected to finite deformation under hygrothermoelastic theory were examined by Advani et al. [20]. A coupled micro-macro mechanical approach was adopted by Aboudi and Williams [21] to examine hygrothermoelastic composites. Altay and Dokmeci [22] derived Hamiltonian-type variational principles for governing the behavior of a hygrothermoelastic medium. Rao and Sinha’s [23] research analyzes how multidirectional composites behave in a hygrothermal medium. Vibration characteristics of hygrothermoelastic laminated composite doubly curved shells were studied by Kundu and Han [24]. Chiba and Sugano [25] discussed how layered plates behave under the influence of hygrothermal loading. Alsubari et al. [26] studied on the bending behavior of simply supported anisotropic cylindrical shells under the influence of moisture and temperature. The bending of simply supported orthotropic cylindrical shells under hygrothermoelasticity was demonstrated by Mohamed et al. [27]. Hosseini and Ghadiri [28] discussed two-dimensional problems in a coupled hygrothermoelastic medium. The potential theory method was adopted by Zhao et al. [29] to obtain a steady-state solution in a hygrothermoelastic medium. Zhang et al. [30] solved the time-fractional hygrothermoelastic problem of a centrally symmetric sphere. Lamba and Deshmukh [31] discussed the unsteady state responses of a finite long solid cylinder subjected to axisymmetric hygrothermal loading. Bhoyar et al. [32] employed a two-temperature model for hygrothermoelastic diffusion theory and discussed the bending of an elliptic plate. Recently Ailawalia et al. [33] discussed wave propagation in an initially stressed hygrothermoelastic medium. Anand et al. [34] investigated the reflection of hygrothermal waves in a nonlocal theory of coupled thermo-elasticity. In addition to the above-mentioned work, some other prominent work [35-41] has been done in the field of thermoelasticity.

This paper deals with the study of deformation in a hygrothermoelastic rod of length $l$. The rod is subjected to sudden heating at one end. The displacement components, moisture concentration, temperature distribution, and stress components are evaluated and presented graphically to show the effect of these quantities at different time intervals.

2. Basic equations

A thin semi-infinite hygrothermoelastic rod is considered. The rod occupies the region $x \geq 0$. Following Hosseini and Ghadiri [26], governing equations in the hygrothermoelastic medium without body forces and heat sources are given by:

$$\sigma_{ji,j} = \rho \ddot{u}_i,$$  \hspace{1cm} (2.1)
\[ D_{T} \theta_{ij} + D_{m}^{T} m_{ij} - \dot{\theta} - \frac{\alpha_{ij} T_{0}}{\rho c} u_{j,j} = 0, \]  
\[ D_{m} m_{ij} + D_{m}^{T} \theta_{ij} - \dot{m} - \frac{\alpha_{ij} m_{0} D_{m}}{k_{m,0}} u_{j,j} = 0, \]

where

\[ \beta_{ij}^{T} = \alpha_{T} \delta_{ij}, \quad \alpha_{T} = (3\lambda + 2\mu) \gamma_{T}, \]  
\[ \beta_{ij}^{m} = \alpha_{m} \delta_{ij}, \quad \alpha_{m} = (3\lambda + 2\mu) \gamma_{m}. \]  

Further, constitutive stress-strain relations [26] are given by:

\[ \sigma_{ij} = C_{ijkl} e_{ij} - \beta_{ij}^{m} m - \beta_{ij}^{T} \theta, \]  

where

\[ C_{ijkl} = \frac{2G\nu}{l - 2\nu} \delta_{ij} \delta_{kl} + G\delta_{ik} \delta_{jl} + G\delta_{il} \delta_{jk}, \]  
\[ c_{ij} = \frac{u_{j,j} + u_{i,j}}{2}. \]

The displacement components for the problem under consideration are assumed to be of the form \( u_{x} = u(x,t), \) \( u_{y} = u_{z} = 0. \)

For the considered one-dimensional problem Eqs (2.1), (2.2), and (2.3) reduce to,

\[ \left( \lambda + 2\mu \right) \frac{\partial^{2} u}{\partial x^{2}} - \beta_{m} \frac{\partial m}{\partial x} - \beta_{T} \frac{\partial \theta}{\partial x} = \rho \frac{\partial^{2} u}{\partial t^{2}}, \]  
\[ D_{T} \frac{\partial^{2} \theta}{\partial x^{2}} + D_{m}^{T} \frac{\partial^{2} m}{\partial x^{2}} - \frac{\partial \theta}{\partial t} - \frac{\beta_{T} T_{0}}{\rho c} \frac{\partial u}{\partial x} = 0, \]  
\[ D_{m} \frac{\partial^{2} m}{\partial x^{2}} + D_{m}^{T} \frac{\partial^{2} \theta}{\partial x^{2}} - \frac{\partial m}{\partial t} - \frac{\beta_{m} m_{0} D_{m}}{k_{m,0}} \frac{\partial u}{\partial x} = 0. \]

The stress component \( \sigma_{xx} \) in one dimension reduces to,

\[ \sigma_{xx} = \left( \lambda + 2\mu \right) \frac{\partial u}{\partial x} - \beta_{m} m - \beta_{T} \theta. \]  

Introducing the following dimensionless variables,

\[ x' = \frac{l}{x}, \quad u' = \frac{l}{u}, \quad \theta' = \frac{D_{m}}{l^{2}} t, \quad m' = m, \quad \theta' = \frac{\theta}{T_{0}}, \quad \sigma_{xx}' = \frac{\sigma_{xx}}{\lambda}. \]
in Eqs (2.9)-(2.11), we get the following equations in dimensionless form,

\[
(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \beta_m \frac{\partial m}{\partial x} - \beta_T T_0 \frac{\partial \theta}{\partial x} = \frac{\rho D_m^2}{l^2} \frac{\partial^2 u}{\partial t^2},
\]

(2.14)

\[
D_T T_0 \frac{\partial^2 \theta}{\partial x^2} + D_m \frac{\partial^2 m}{\partial x^2} - D_m T_0 \frac{\partial}{\partial t} \left( \beta_T T_0 \frac{\partial}{\partial x} \right) = 0,
\]

(2.15)

\[
D_m \frac{\partial^2 m}{\partial x^2} + D_m T_0 \frac{\partial^2 \theta}{\partial x^2} - D_m \frac{\partial}{\partial t} \left( \beta m m_0 D_m^2 \frac{\partial}{\partial x} \right) = 0.
\]

(2.16)

The dimensionless stress component \( \sigma_{xx} \) is given by,

\[
\sigma_{xx} = \frac{1}{\lambda \left( \lambda + 2\mu \right)} \frac{\partial u}{\partial x} - \beta m m - \beta_T T_0 \theta.
\]

(2.17)

3. Solution

The solution of the physical variables may be assumed in the form of normal modes as:

\[
\{ u, m, \theta \} = \left[ \bar{u}, \bar{m}, \bar{\theta} \right](x) \exp(\omega t),
\]

(3.1)

where \( \omega \) is complex frequency.

Using the above solution in (2.14)-(2.16), we obtain the following equations:

\[
\left[ a_1 D^2 - a_2 \right] \bar{u} - a_3 D \bar{m} - a_4 D \bar{\theta} = 0,
\]

(3.2)

\[
b_1 D \bar{u} + b_2 D^2 \bar{m} + \left[ b_3 D^2 - b_4 \right] \bar{\theta} = 0,
\]

(3.3)

\[
c_1 D \bar{u} + \left[ c_2 D^2 - c_3 \right] \bar{m} + c_4 D^2 \bar{\theta} = 0,
\]

(3.4)

where

\[
a_1 = \lambda + 2\mu, \quad a_2 = \frac{\rho c^2 \omega^2}{l^2}, \quad a_3 = \beta m, \quad a_4 = \beta_T T_0,
\]

\[
b_1 = \frac{-a_2 c_4 \omega^2}{\rho c}, \quad b_2 = D_m^2, \quad b_3 = D_T T_0, \quad b_4 = c_2 T_0 \omega,
\]

(3.5)

\[
c_1 = \frac{-\beta m m_0 c_4^3 \omega}{k_m}, \quad c_2 = D_m, \quad c_3 = c_2 \omega, \quad c_4 = D_m T_0, \quad D = \frac{d}{dx}.
\]

On solving Eqs (3.2)-(3.4), we obtain a sixth-order differential equation in terms of \( \bar{u}, \bar{m}, \bar{T} \) as:
\[
[D^6 + A_1 D^4 + A_2 D^2 + A_3] (\bar{u}, \bar{m}, \bar{\theta}) = 0,
\]
where
\[
A_1 = \frac{f_1 g_5 - f_2 g_4 + f_3 g_1 - f_3 g_2}{f_1 g_4 - f_3 g_1}, \quad A_2 = \frac{f_4 g_2 - f_3 g_3 - f_2 g_5}{f_1 g_4 - f_3 g_1}, \quad A_3 = \frac{f_4 g_3}{f_1 g_4 - f_3 g_1},
\]
\[
f_1 = a_1 c_4, \quad f_2 = a_2 c_4 - a_3 c_1, \quad f_3 = a_4 c_2 - a_3 c_4, \quad f_4 = a_4 c_3,
\]
\[
g_1 = a_1 b_3, \quad g_2 = a_2 b_1 - a_3 b_4 - a_2 b_3, \quad g_3 = a_2 b_4, \quad g_4 = a_4 b_2 - a_3 b_3, \quad g_5 = a_1 b_4.
\]

The appropriate solution of Eq.(3.6) satisfying the radiation conditions may be expressed as:
\[
\bar{u} = B_1 \exp\{-k_1 x\} + B_2 \exp\{k_1 x\} + B_3 \exp\{-k_2 x\} + \ldots
+ B_4 \exp\{k_2 x\} + B_5 \exp\{-k_3 x\} + B_6 \exp\{k_3 x\},
\]
\[
\bar{m} = B_1^* \exp\{-k_1 x\} + B_2^* \exp\{k_1 x\} + B_3^* \exp\{-k_2 x\} + \ldots
+ B_4^* \exp\{k_2 x\} + B_5^* \exp\{-k_3 x\} + B_6^* \exp\{k_3 x\},
\]
\[
\bar{\theta} = B_1^{**} \exp\{-k_1 x\} + B_2^{**} \exp\{k_1 x\} + B_3^{**} \exp\{-k_2 x\} + \ldots
+ B_4^{**} \exp\{k_2 x\} + B_5^{**} \exp\{-k_3 x\} + B_6^{**} \exp\{k_3 x\},
\]
where
\[
\pm k_i^2 (i = 1, 2, 3)
\]
are the roots of
\[
(k^6 + A_1 k^4 + A_2 k^2 + A_3) = 0,
\]
and the parameters \(B_j^*\) and \(B_j^{**}\) depend on \(B_j,(j = 1, \ldots, 6)\) and are given by:
\[
B_1^* = H_{11} B_1, \quad B_2^* = -H_{12} B_2, \quad B_3^* = H_{12} B_3, \quad B_4^* = -H_{12} B_4,
\]
\[
B_5^* = H_{13} B_5, \quad B_6^* = -H_{13} B_6, \quad B_1^{**} = -H_{21} B_1, \quad B_2^{**} = H_{21} B_2,
\]
\[
B_3^{**} = -H_{22} B_3, \quad B_4^{**} = H_{22} B_4, \quad B_5^{**} = -H_{23} B_5, \quad B_6^{**} = H_{23} B_6,
\]
\[
H_{11} = \frac{k_i \left( f_1 k_i^2 - f_2 \right)}{f_3 k_i^2 - f_4}, \quad H_{21} = \frac{a_1 k_i^2 - a_2 + a_5 k_i H_{11}}{a_4 k_i},
\]

4. Boundary conditions

We consider the stress-free hygrothermoelastic rod at a uniform temperature \(T_0\) with its boundary \(0 \leq x \leq l\). The boundary \(x = 0\) is subjected to sudden heating. The appropriate boundary conditions are given by,
\( u(0,t) = 0 \),

\( u(l,t) = 0 \),

\( \frac{\partial m}{\partial x}(0,t) = 0 \),

\( \frac{\partial m}{\partial x} = 0 \),

\( \theta(0,t) = \theta_0 \exp(\omega t) \),

\( \theta(l,t) = 0 \).

Using \((3.1), (3.8)-(3.10)\) in the boundary conditions \((4.1)\), we get the following non-homogenous system of six equations:

\[
B_1 + B_2 + B_3 + B_4 + B_5 + B_6 = 0, \tag{4.2}
\]

\[
B_1\exp\{-k_1l\} + B_2\exp\{k_1l\} + B_3\exp\{-k_2l\} + B_4\exp\{k_2l\} + B_5\exp\{-k_3l\} + B_6\exp\{k_3l\} = 0, \tag{4.3}
\]

\[
k_1H_1B_1 + k_1H_1B_2 + k_1H_1B_3 + k_2H_1B_4 + k_2H_1B_5 + k_2H_1B_6 = 0, \tag{4.4}
\]

\[
k_1H_2\exp\{-k_1l\}B_1 + k_1H_2\exp\{k_1l\}B_2 + k_2H_2\exp\{-k_2l\}B_3 + k_2H_2\exp\{k_2l\}B_4 + k_2H_2\exp\{-k_3l\}B_5 + k_2H_2\exp\{k_3l\}B_6 = 0, \tag{4.5}
\]

\[
-H_2B_1 + H_2B_2 - H_2B_3 + H_2B_4 - H_2B_5 + H_2B_6 = \theta_0 \exp(\omega t), \tag{4.6}
\]

\[
-H_3\exp\{-k_1l\}B_1 + H_3\exp\{k_1l\}B_2 - H_3\exp\{-k_2l\}B_3 + H_3\exp\{k_2l\}B_4 - H_3\exp\{-k_3l\}B_5 + H_3\exp\{k_3l\}B_6 = 0. \tag{4.7}
\]

The above non-homogenous system of six equations is solved by developing codes in MATLAB and the values of constants \( B_n \) \((n = 1, \ldots, 6)\) are evaluated.

Using the expressions of \( \bar{\pi}, \bar{m} \) and \( \bar{\theta} \) given by \((3.8)-(3.10)\) in expressions \((3.1)\), the displacement component, moisture concentration, temperature field, and stress in hygrothermoelastic medium are obtained.

5. Numerical computation

The analytical results are verified in this section by taking a wood slab as a hygro-thermoelastic material. The elastic constants given by Chang and Weng [42] and Yang et al. [43] are shown in Tab.1.
Table 1. Values of material constants.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$64.3 \times 10^9 \text{ N/m}^2$</td>
<td>$k$</td>
<td>$0.65 \text{ w/m} (\circ \text{K})$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$370 \text{ Kg/m}^3$</td>
<td>$c$</td>
<td>$2500 \text{ J/Kg} (\circ \text{K})$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$0.33$</td>
<td>$K_m$</td>
<td>$2.2 \times 10^{-8} \text{ Kg/msM}$</td>
</tr>
<tr>
<td>$m_0$</td>
<td>$10%$</td>
<td>$D_m$</td>
<td>$2.16 \times 10^{-6} \text{ m}^2/\text{sec}$</td>
</tr>
<tr>
<td>$\alpha_m$</td>
<td>$2.68 \times 10^{-3} \text{ cm/cm}(%\text{H}_2\text{O})$</td>
<td>$D_m^\alpha$</td>
<td>$0.648 \times 10^{-6} \text{ m}^2/(%\text{H}_2\text{O})/s(\circ \text{K})$</td>
</tr>
<tr>
<td>$T_0$</td>
<td>$283^\circ \text{K}$</td>
<td>$D_m^\alpha$</td>
<td>$2.1 \times 10^{-7} \text{ m}^2/(\circ \text{K})/s(%\text{H}_2\text{O})$</td>
</tr>
<tr>
<td>$\alpha^T$</td>
<td>$31.3 \times 10^{-6} \text{ cm/cm}(\circ \text{K})$</td>
<td>$D_T$</td>
<td>$k/\rho c$</td>
</tr>
</tbody>
</table>

The numerical results are obtained for displacement, force stress, moisture concentration, and temperature distribution for $l=1.0$, $\theta_0=1.0$ against the horizontal distance $x$. The graphical results are shown for four values of non-dimensional time ($t=0.01$, $0.1$, $0.5$ and $1.5$).

6. Discussions

The variations of displacement follow a linear trend in the region $0 \leq x < 8.2$ and then increase sharply. This sharpness is maximum for $t=0.01$ and more interestingly, the sharpness decreases with an increase in time $t$, respectively. This behavior in the region $0 \leq x < 8.2$ can be attributed to the material's elastic response, where the displacement is directly proportional to the applied stress. The sudden increase can be attributed to the occurrence of a stress concentration or a localized deformation mechanism. It is possible that the material reaches its limit of linear elasticity and undergoes plastic deformation or experiences a structural discontinuity. The variations of force stress are linear in nature with differences in slope. Also, the slope of the variation of force stress decreases with an increase in time $t$. The difference in slopes suggests that different regions of the material experience different levels of stress, which can be attributed to varying mechanical properties, geometric constraints, or external loading conditions.

Similar to the variations of force stress, the variations of moisture concentration also follow a linear path. It is to mention here that the slope of the variation of moisture concentration is zero, i.e., the variations are parallel to the horizontal distance. However, the magnitude of the values decreases with an increase in time. The parallel nature of the variations (i.e., zero slope) suggests that the moisture concentration does not change along the horizontal distance, implying a constant moisture gradient. This behavior could be influenced by factors such as moisture diffusion, capillary action, or moisture transport through the material. The variations in the temperature field are similar in nature to the variations of displacement. These variations of displacement, force stress, moisture concentration, and temperature field are shown in Figs 1-4, respectively.
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Fig. 1. Displacement with distance $x$.

Fig. 2. Force stress with distance $x$. 
7. Conclusion

The analytical and numerical findings of the research problem are as follows:
The results can help in the design and optimization of materials and structures for various engineering applications, where thermal and moisture-induced stresses can lead to failures or reduced performance. The analytical results show that three waves propagate in the medium namely displacement, moisture, and thermal wave. The variations of force stress and moisture concentration are linear in nature. The variations of displacement and temperature field are similar in nature. As expected, the moisture concentration remains constant with horizontal distance.

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Nomenclature

- $\epsilon_{rs}$ – components of strain
- $\sigma_{ij}$ – components of stress
- $u_i$ – components of displacement
- $\rho$ – density
- $D_T$ – temperature diffusivity
- $\theta$ – temperature field
- $m$ – moisture concentration
- $T_0$ – initial temperature
- $D_m$ – moisture diffusivity
- $D_T^T$, $D_m^T$ – diffusivities
- $c$ – heat capacity
- $m_0$ – reference moisture
- $k_m$ – moisture diffusivity
- $\alpha_{ij}^T$ and $\alpha_{ij}^m$ – material coefficients
- $\lambda, \mu$ – Lame's constants such that $\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$ and $\mu = \frac{E}{2(1 + \nu)}$

References


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