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Brief note

PROJECTIVE SYNCHRONIZATION OF CHAOTIC SYSTEMS VIA BACKSTEPPING DESIGN

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Chaos synchronization of discrete dynamical systems is investigated. An algorithm is proposed for projective synchronization of chaotic 2D Duffing map and chaotic Tinkerbell map. The control law was derived from the Lyapunov stability theory. Numerical simulation results are presented to verify the effectiveness of the proposed algorithm.

Key words: Lyapunov function, projective synchronization, backstepping, design.

1. Introduction

Adjacent chaotic trajectories governed by the same nonlinear systems may evolve into a state utterly uncorrelated, but in 1990 Pecora and Carrol showned that it could be synchronized through a proper coupling.

Since their paper in 1990, chaos synchronization is an interesting research topic of great attention. Hayes *et al*. (1993) studied some potential applications in secure communication , Blasius *et al*. observed complex dynamics and phase synchronization in spatially extended ecological systems in 1999 and system identification was investigated by Kocarev *et al*. (1996). Different forms of synchronization phenomena have been observed in a variety of chaotic systems, such as identical synchronization (Pecora and Carroll, 1990). Rosenblum *et al*. (1996) studied phase synchronization of chaotic oscillators. A generalized synchronization for unidirectionally coupled dynamical systems was proposed by Rulkov *et al*. (1995), where two systems are called synchronized if a static functional relation exists between the states of the systems. The linear generalized chaos synchronization and predictability were investigated by Poria (2007). Recently, Khan *et al*. (2009) studied generalized chaos synchronization of a coupled Rossler system. A generalized anti-synchronization of different chaotic systems was investigated by Khan *et al*. (2012). Tarai *et al*. (2007) studied synchronization of a bi-directionally coupled chaotic unified Chen's system with delay and they also studied generalized synchronization of a linearly bi-directionally coupled unified chaotic system in 2009.

Along with the development of control technology (Khan *et al*., 2011), the requirement for control precision gets higher and higher. In order to achieve the satisfied control performance, nonlinear control techniques are used. Backstepping is one of these nonlinear control techniques that has attracted a great deal of research interest in the last few years. Yongguang *et al*. (2003) studied controlling uncertain Lu system using backstepping design. In 2001, adaptive synchronization of uncertain chaotic systems via backstepping

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design was investigated by Wang *et al*. Tan *et al*. (2003) applied backstepping design for synchronizing of chaotic systems. Chaos control of 4D chaotic systems using recursive backstepping nonlinear controller was investigated by Laoye *et al*. (2007).

The key idea of the backstepping approach is to recursively design controllers for subsystems in the structure and "step back" the feedback signals towards the control input. This differs from the conventional feedback linearization in that it offers a more flexible way of dealing with uncertainties. Using the Lyapunov functions, the system can be taken into account and harmful nonlinearities can be cancelled or dominated by the control signal.

Gonzalez-Miranda (1998) derived a general condition for projective synchronization (PS) in arbitrary dimensional systems. In projective synchronization the drive and response vectors synchronize up to a scaling factor. PS is characterized by a scale factor that defines a proportional relation between the synchronized systems. PS results from the partial linearity of coupled chaotic systems, and becomes the unique feature of partially linear systems, studied by Maineeri and Rehacek (1999). The proportionality makes it possible to duplicate a chaotic system with different scales, while the topological characteristic of the two synchronized systems remain unchanged.

2. Backstepping design for Duffing map

We consider the drive Duffing map as

$$\begin{aligned} x_1(k+1) &= x_2(k), \\ x_2(k+1) &= -bx_1(k) + ax_2(k) - x_2^3(k), \end{aligned} \tag{2.1}$$

and the driven Duffing map as

$$\begin{aligned} y_1(k+1) &= y_2(k), \\ y_2(k+1) &= -by_1(k) + ay_2(k) - y_2^3(k) + u(k). \end{aligned} \tag{2.2}$$

Here a and b are the parameters of the Duffing map. The uncontrolled Duffing map displays chaos for $a = 2.75$ and $b = 0.2$. Now our aim is to find $u(k)$ such that the drive and response vectors become proportional, i.e., $\lim_{n\to\infty} \frac{y_i(n)}{x_i(n)} = \alpha$. Defining the errors as $e_1(k) = \alpha y_1(k) - x_1(k)$, $e_2(k) = \alpha y_2(k) - x_2(k)$ we obtain the dynamical system for the error as

$$\begin{aligned} e_1(k+1) &= e_2(k), \\ e_2(k+1) &= -be_1(k) + ae_2(k) - \alpha y_2^3(k) + x_2^3(k) + \alpha u(k). \end{aligned} \tag{2.3}$$

Now the systems Eqs (2.1) and (2.2) will synchronize if for a suitable choice of $u(k)$ the error system stabilizes at the origin.

Following the backstepping technique we assume $z_1(k) = e_1(k)$ and choose the Lyapunov function $V_1(k) = |e_1(k)|$. Then the variation of $V_1(k)$ with time is

$$\begin{aligned}
\Delta V_1(k) &= V_1(k+1) - V_1(k) \\
&= |e_1(k+1)| - |e_1(k)| \\
&= |e_2(k)| - |z_1(k)| \\
&= |e_2(k) - z_2(k) + z_2(k)| - |z_1(k)| \\
&= |\beta(k) + z_2(k)| - |z_1(k)| \ \text{ introducing } z_2(k) = e_2(k) - \beta(k) \\
&= |c_1 z_1(k) + z_2(k)| - |z_1(k)| \ \text{ choosing } \beta(k) = c_1 z_1(k) \\
&\le \big||z_2(k)| + (|c_1| - 1) z_1(k)\big|
\end{aligned} \tag{2.4}$$

where c_1 is a design constant to be chosen later.

Now

$$z_1(k+1) = e_1(k+1) = e_2(k) = z_2(k) + \beta(k) = c_1 z_1(k) + z_2(k). \tag{2.5}$$

Again,

$$\begin{aligned}
z_2(k+1) &= e_2(k+1) - \beta(k+1) = \\
&= -b e_1(k) + a e_2(k) - \alpha y_2^3(k) + x_2^3(k) + \alpha u(k) - c_1\big(c_1 z_1(k) + z_2(k)\big).
\end{aligned} \tag{2.6}$$

At this step we can determine the control law $u(k)$ as

$$u(k) = \Big[c_2 z_2(k) + b e_1(k) - a e_2(k) + \alpha y_2^3(k) - x_2^3(k) + c_1\big(c_1 z_1(k) + z_2(k)\big)\Big]\Big/\alpha \tag{2.7}$$

where c_2 is another design constant to be choosen later. Now with this $u(k)$ Eq.(2.7) becomes

$$z_2(k+1) = c_2 z_2(k). \tag{2.8}$$

Choosing the second Lyapunov function as $V_2(k) = V_1(k) + d|z_2(k)|$, where d is a positive constant, the variation of $V_2(k)$ can be computed as

$$\begin{aligned}
\Delta V_2(k) &= V_2(k+1) - V_2(k) \\
&= V_1(k+1) + d|z_2(k+1)| - V_1(k) - d|z_2(k)| \\
&= \Delta V_1(k) + d|z_2(k+1)| - d|z_2(k)| \\
&\le |z_2(k)| + (|c_1| - 1)|z_1(k)| + d|c_2||z_2| - d|z_2(k)| \\
&= (|c_1| - 1)|z_1(k)| + (d|c_2| - d + 1)|z_2(k)|.
\end{aligned} \tag{2.9}$$

We choose c_1 and c_2 such that the right hand side of Eq.(2.9) becomes negative definite. Clearly, the right hand side of Eq.(2.9) is negative definite if we choose $|c_1| < 1$ and $|c_2| < \frac{d-1}{d}$ where $d > 1$. We can choose the parameter d to be large enough. Hence the close loop system Eqs.(2.5) and (2.8)

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{pmatrix} c_1 & 1 \\ 0 & c_2 \end{pmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = A \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \tag{2.10}$$

is globally stable about the origin. Therefore system Eq.(2.1) is synchronized with system Eq.(2.2) up to a constant scaling factor α. Notice that if we choose any one of the design constants c_1, c_2 to be zero, the dead-beat response will happen since $A^2 = 0$, which implies that the error state will be brought to zero in at most 2 time steps. Therefore projective synchronization in finite time can be achieved.

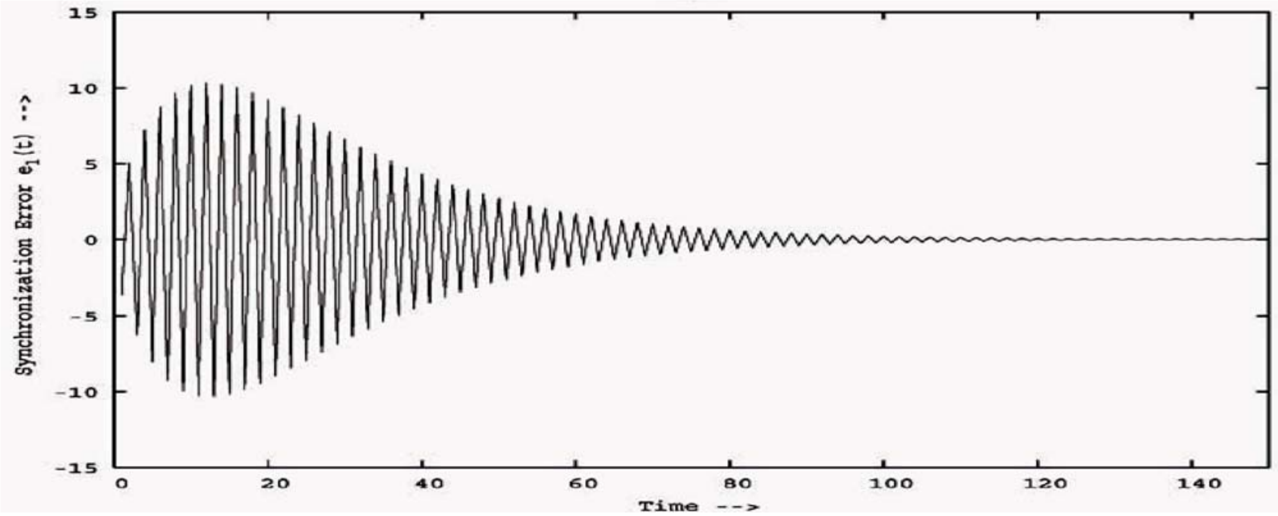

Fig.1. Time evolution of the synchronization error $e_1(t)$ for $c_1 = -0.9$, $c_2 = -0.95$ for Duffing map.

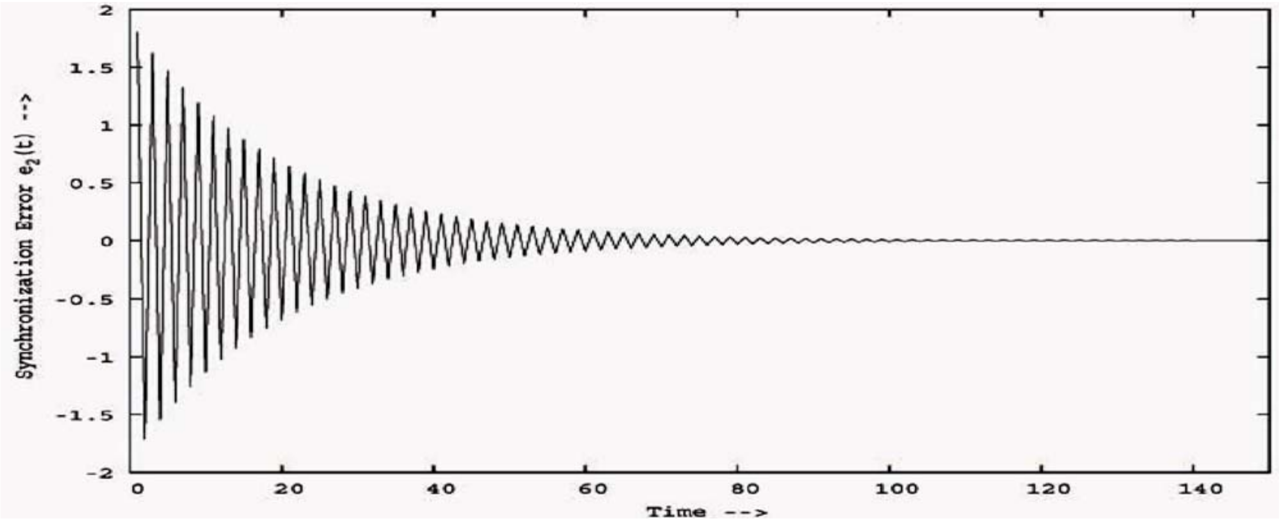

Fig.2. Time evolution of the synchronization error $e_2(t)$ for $c_1 = -0.9$, $c_2 = -0.95$ for Duffing map.

3. Backstepping design for Tinkerbell map

We consider the driving Tinkerbell map as

$$\begin{aligned} x_1(k+1) &= x_1^2(k) - x_2^2(k) + ax_1(k) + bx_2(k), \\ x_2(k+1) &= 2x_1(k)x_2(k) + cx_1(k) + dx_2(k), \end{aligned} \tag{3.1}$$

and the driven Tinkerbell map as

$$\begin{aligned} y_1(k+1) &= y_1^2(k) - y_2^2(k) + ay_1(k) + by_2(k), \\ y_2(k+1) &= 2y_1(k)y_2(k) + cy_1(k) + dy_2(k) + u(k). \end{aligned} \tag{3.2}$$

Here a, b, c, d are parameters of the map. The uncontrolled map displays chaos for $a = 0.9, b = 0.6013$, $c = 2$ and $d = 0.5$. Now our aim is to find a controller $u(k)$ such that the drive and response system synchronize projectively. Defining the errors as $e_1(k) = \alpha y_1(k) - x_1(k)$, $e_2(k) = \alpha y_2(k) - x_2(k)$ we obtain the dynamical system for the error as

$$\begin{aligned} e_1(k+1) &= ae_1(k) + be_2(k) + \alpha\left(y_1^2(k) - y_2^2(k)\right) - \left(x_1^2(k) - x_2^2(k)\right), \\ e_2(k+1) &= ce_1(k) + d_2(k) + 2\alpha\left(y_1(k)y_2(k) - x_1(k)x_2(k)\right) + \alpha u(k). \end{aligned} \tag{3.3}$$

For synchronization of Eqs (3.1) and (3.2) the error system should be stabilized at the origin. According to backstepping design we assume $z_1(k) = e_1(k)$. Choosing the Lyapunov function as $V_1(k) = |e_1(k)|$, we get the time variation of $V_1(k)$ as

$$\begin{aligned} &\Delta V_1(k) = \left|e_1(k+1)\right| - \left|e_1(k)\right| = \\ &= \left|\alpha\left(y_1^2(k) - y_2^2(k)\right) - \left(x_1^2(k) - x_2^2(k) + ae_1(k) + be_2(k)\right| - \left|z_1(k)\right)\right|. \end{aligned} \tag{3.4}$$

We choose

$$\beta(k) = c_1 z_1(k) - \alpha\left(y_1^2(k) - y_2^2(k)\right) - \left(x_1^2(k) - x_2^2(k) = c_1 z_1(k) + \varphi(k)\right)(\text{say}), \tag{3.5}$$

here c_1 is a constant to be chosen later. We introduce the variable $z_2(k) = e_2(k) - \beta(k)$, now,

$$\Delta V_1(k) = \left|\alpha\left(y_1^2(k) - y_2^2(k)\right) - \left(x_1^2(k) - x_2^2(k)\right) + az_1(k) + be_2(k) + z_2(k) - z_2(k)\right| - \left|z_1(k)\right|$$

$$\leq \left|\alpha\left(y_1^2(k) - y_2^2(k)\right) - \left(x_1^2(k) - x_2^2(k)\right) + be_2(k) + z_2(k) - z_2(k)\right| + (a-1)\left|z_1(k)\right|$$

$$
\begin{aligned}
&=\left|\beta(k)+z_2(k)+\alpha\left(y_1^2(k)-y_2^2(k)\right)-\left(x_1^2(k)-x_2^2(k)\right)\right|+(a-1)\left|z_1(k)\right| \\
&=\left|c_1 z_1(k)+z_2(k)\right|+(a-1)\left|z_1(k)\right| \\
&\le\left|z_2(k)\right|+\left(\left|c_1\right|+a-1\right)\left|z_1(k)\right|.
\end{aligned}
\tag{3.6}
$$

Therefore

$$
\begin{aligned}
z_1(k+1)&=e_1(k+1)=\alpha\left(y_1^2(k)-y_2^2(k)\right)-\left(x_1^2(k)-x_2^2(k)\right)+ae_1(k)+be_2(k) \\
&=\alpha\left(y_1^2(k)-y_2^2(k)\right)-\left(x_1^2(k)-x_2^2(k)\right)+az_1(k)+z_2(k)+\beta(k) \\
&=c_1 z_1(k)+az_1(k)+z_2(k) \\
&=\left(c_1+a\right)z_1(k)+z_2(k).
\end{aligned}
\tag{3.7}
$$

Now we have

$$
\begin{aligned}
z_2(k+1)&=be_2(k+1)-\beta(k+1)= \\
&=b\left[2\alpha\left(y_1(k)y_2(k)-x_1(k)x_2(k)\right)+ce_1(k)+de_2(k)+\alpha u(k)\right]+ \\
&-c_1\left[\left(c_1+a\right)z_1(k)+z_2(k)\right]-\varphi(k+1).
\end{aligned}
\tag{3.8}
$$

At this step we choose the control law $u(k)$ as

$$
\begin{aligned}
u(k)=&\left[c_2 z_2(k)-2b\alpha\left(y_1(k)y_2(k)-x_1(k)x_2(k)\right)+\right. \\
&\left.-cbe_1(k)-bde_2(k)+\left\{c_1\left(c_1+a\right)z_1(k)+z_2(k)\right\}\right]+\varphi(k+1)/b\alpha
\end{aligned}
\tag{3.9}
$$

With this choice we get

$$
z_2(k+1)=c_2 z_2(k).
\tag{3.10}
$$

Let us take the second Lyapunov function as $V_2(k)=V_1(k)+d\left|z_2(k)\right|$ where d is a positive constant. Then the time variation of $V_2(k)$ is given by

$$
\begin{aligned}
&\Delta V_2(k)=V_2(k+1)-V_2(k) \\
&=\Delta V_1(k)+d\left|z_2(k+1)\right|+d\left|z_2(k)\right| \\
&\le\left|z_2(k)\right|+\left(\left|c_1\right|+a-1\right)\left|z_1(k)\right|+d\left|c_2\right|\left|z_2(k)\right|-d\left|z_2(k)\right| \\
&=\left(\left|c_1\right|+a-1\right)\left|z_1(k)\right|+\left(d\left|c_2\right|-d+1\right)\left|z_2(k)\right|.
\end{aligned}
\tag{3.11}
$$

In order to make the right hand side of Eqs (3.11) negative definite we have to choose $a-1<c_1<1-a, \left|c_2\right|<\frac{d-1}{d}$ where $d>1$. Now the close-loop system (3.7) and (3.10)

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{pmatrix} c_1+a & 1 \\ 0 & c_2 \end{pmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = A \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \tag{3.12}$$

is globally stable about the origin. Therefore system Eq.(2.1) is synchronized with system Eq.(2.2) up to a constant scaling factor α. Notice that if we choose any one of the design constants $c_1=-a,$ or c_2 to be zero, the dead-beat response will happen since $A^2=0$, which implies that the error state will be brought to zero in at most 2 time steps. Therefore projective synchronization in finite time can be achieved.

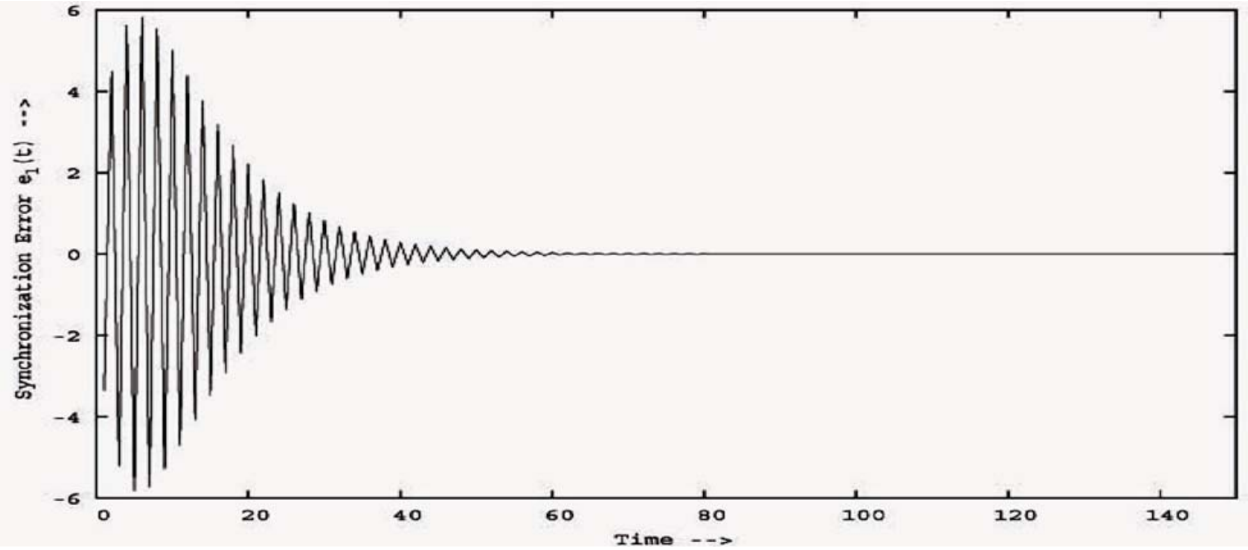

Fig.3. Time evolution of the synchronization error $e_1(t)$ for $c_1=-1.7$, $c_2=-0.9$ for Tinkerbell map.

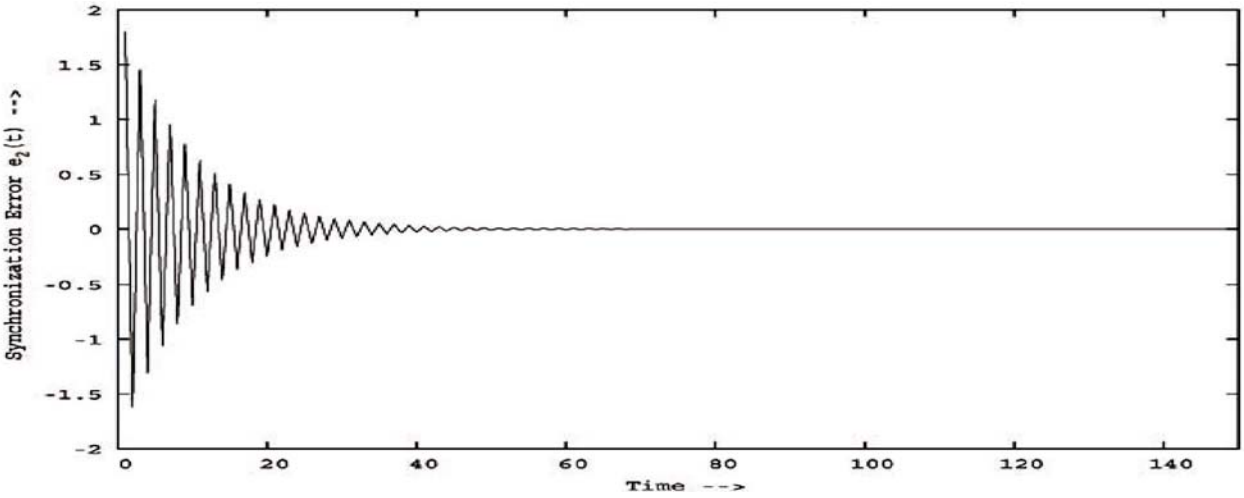

Fig.4. Time evolution of the synchronization error $e_2(t)$ for $c_1=-1.7$, $c_2=-0.9$ for Tinkerbell map.

4. Results

We have successfully applied the backstepping method for projective synchronization of the Duffing map and Tinkerbell map. The numerical simulation results for the case of the Duffing map are shown in Fig.1 and Fig.2 taking the value of $c_1 = -0.9$ and $c_2 = -0.95$. In both the cases the error goes to zero through oscillation. We assume the synchronization errors were initially $e_1(t) = 1.9$ and $e_2(t) = -1.9$. Simulation results for the case of the Tinkerbell map are shown in Fig.3 and Fig.4 taking the values of $c_1 = -1.7$ and $c_2 = -0.9$. Here also the error goes to zero in an oscillatory way. The numerical simulation results indicate that this approach works very well.

Nomenclature

PS – projective synchronization
α – scaling factor
$\Delta V_1(k)$ – variation of $V_1(k)$

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