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Brief note

MODEL OF 5R SPATIAL LINKAGE IN THE GEOMETRY OF TORI

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The present study covers the problem of a kinematic analysis of a spatial 5R single DOF linkage. This mechanism was constructed by combining two 4R universal mechanisms presented in tori geometry (Bil, 2012). The problem can be presented as the effect of an intersection of two tori in such a manner that the circles that form these tori do not coincide (Bil, 2010; 2011; 2012; 2012). As a result of this, a 5R model was obtained of a five-bar mechanism. It was demonstrated that there are 16 cases of the intersection of two universal tori with constant linear dimensions and that they differ with the angles of the mutual positions of rotation axes. A particular case of the five-bar mechanism presented herein is the well-known Goldberg mechanism (Goldberg, 1943), which was initially formed out of a particular case combination of two Bennett mechanisms (Bennett, 1903; Baker, 1979).

Key words: Bennett, Goldberg, spatial mechanism, torus, 5R mechanism.

1. Introduction

The mechanism as proposed in the study (Bil, 2010; 2011; 2012; 2012, Budniak and Bil, 2012) that contains two universal torus surfaces has very wide possibilities. The mechanisms that are equivalent to this system are well-known structures of lever mechanisms with lower pairs. A mechanism with a higher pair in the form of two universal tori is equivalent to the 7R linkage that has warped axes of kinematic pairs.

In the present study, a description is proposed of a model where universal tori intersect in such a manner that those circles that form the bodies of the torus coincide. It was demonstrated in the study by the authors (Budniak and Bil, 2012) that there are two different arrangements of the radiuses of those circles from which identical tori are formed, and each of these arrangements may have two different angular positions of the circles. The universal torus surface is formed by a rotation around the axis of the circle, whose centre is located at a certain distance from this axis, and it does not lie in the plane that passes through this axis.

The position of the circle in relation to the straight line in the space can be defined in two ways (Bil, 2010; 2012; Budniak and Bil, 2012):

- o with the aid of the linear dimension that defines the position of the circle centre at any distance from the straight line and two angles that determine the position of the circle axis,
- o with the aid of two linear dimensions (the distance between the axes and the distance along the axis) and one angle (between the circle axis and the straight line).

For the purpose of the present study, the circles were defined according to the first method.

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2. Spatial four-bar 4r linkage

The manner in which the elements of the mechanism are defined is presented in Figs 1a, b. (Bil, 2010; 2011; 2012; 2012). The base circles were defined with the aid of radiuses r_i (*i=1*, *2*, *3*, *4* – where i is the number of the torus and of its dimensions as well as of the immovable systems of coordinates related to it). The direction versor Z_{i1} of the axis of the circle with the radius s_i is rotated around the axis R_iY_{i1} of the auxiliary system of coordinates $R_iX_{i1}Y_{i1}Z_{i1}$ by the angle χ_i and around the axis R_iY_{i1} by the angle θ_i. By assigning the position of the point S_i on the circle with the aid of the angle ϕ_i, the coordinates of this point are determined in the immovable system of coordinates $Q_iX_iY_iZ_i$. It was accepted that the tori nos. 1 and 3 (Fig.1a) are formed on the basis of the arrangement of radiuses $r_i > s_i$ and those numbered 2 and 4 (Fig.1b) on the basis of radiuses $r_i < s_i$. Tori 1 and 2 as well as 3 and 4 are identical in pairs.

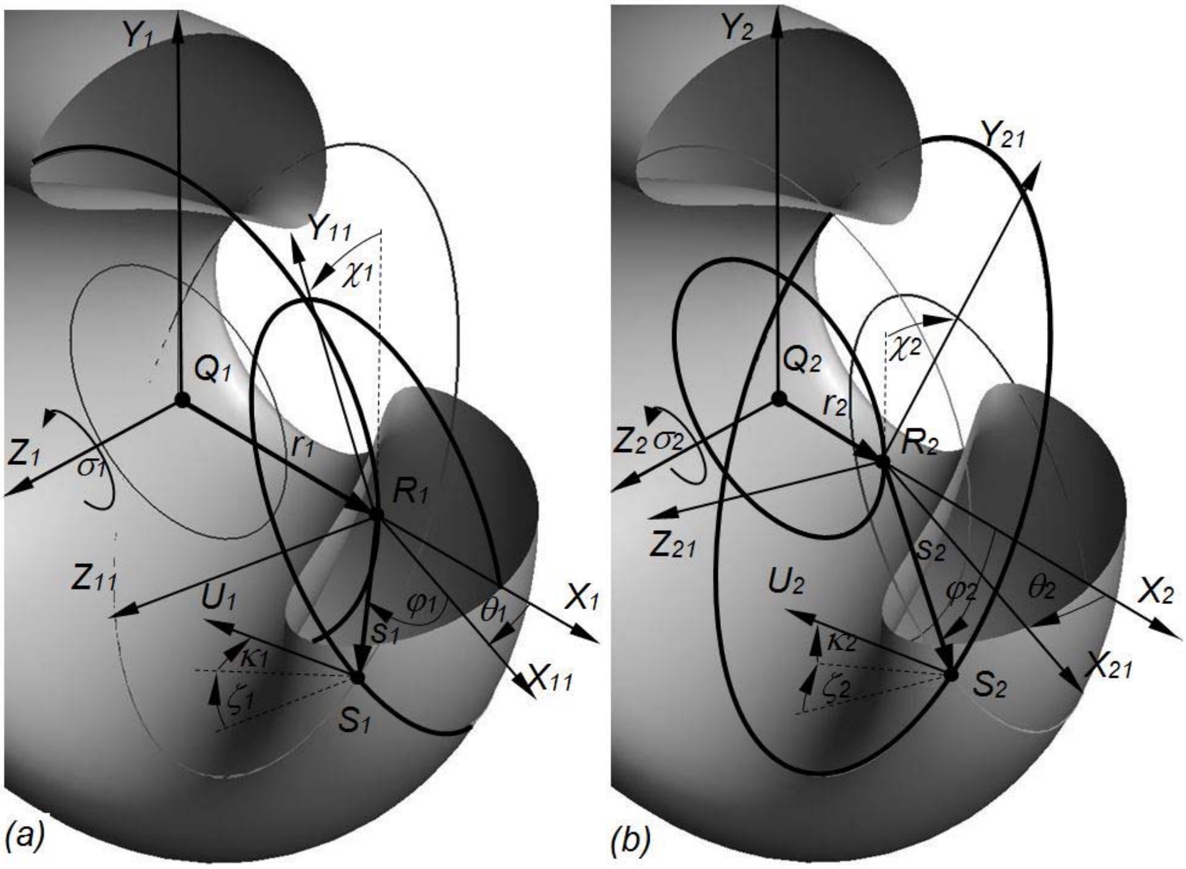

Fig.1. Position of the point Si on the surface of identical generalized tori in the system of coordinates $Q_iX_iY_iZ_i$: a) based on the measurements $r_1 > s_1, \phi_1, \sigma_1, \chi_1, \theta_1$, b) based on the measurements $r_2 < s_2, \phi_2, \sigma_2, \chi_2, \theta_2$.

The point on the surface of the torus can be described in the form of the following vector

$$S_i = \begin{bmatrix} c\sigma_i & -s\sigma_i & 0 \\ s\sigma_i & c\sigma_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi_i & -s\chi_i \\ 0 & s\chi_i & c\chi_i \end{bmatrix} \left\{ \begin{bmatrix} c\theta_i & 0 & s\theta_i \\ 0 & 1 & 0 \\ -s\theta_i & 0 & c\theta_i \end{bmatrix} \begin{bmatrix} c\phi_i & -s\phi_i & 0 \\ s\phi_i & c\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_i \\ 0 \\ 0 \end{bmatrix} \right\} + \begin{bmatrix} r_i \\ 0 \\ 0 \end{bmatrix}, \tag{2.1}$$

which after multiplication leads to

$$S_i = \begin{bmatrix} (s_i c\phi_i c\theta_i + r_i) c\sigma_i - (s_i s\phi_i c\chi_i + s_i c\phi_i s\theta_i s\chi_i) s\sigma_i \\ (s_i c\phi_i c\theta_i + r_i) s\sigma_i + (s_i s\phi_i c\chi_i + s_i c\phi_i s\theta_i s\chi_i) c\sigma_i \\ s_i s\phi_i s\chi_i - s_i c\phi_i s\theta_i c\chi_i \end{bmatrix}. \tag{2.2}$$

Hence, this is the torus surface equation.

A four-bar mechanism including only revolute pairs will be formed when the radiuses r_i and s_i end with revolute pairs, and those versors that determine the direction of the axis of rotation that passes through the common point S will coincide $U_1 = U_2$ or they will be opposite $U_1 = -U_2$ (Bil, 2012; Budniak and Bil, 2012).

The direction of the unit vector U_i (Figs 1a, b) positioned freely in relation to the radius s_i with the beginning at the point S_i in the system of coordinates $Q_i X_i Y_i Z_i$ can be written in the following form

$$U_i = \begin{bmatrix} U_{ix} \\ U_{iy} \\ U_{iz} \end{bmatrix} = \begin{bmatrix} (c\zeta_i s\kappa_i c\phi_i + s\zeta_i s\phi_i) c\theta_i + c\zeta_i c\kappa_i s\theta_i \\ (c\zeta_i s\kappa_i s\phi_i - s\zeta_i c\phi_i) c\chi_i - \left[-(c\zeta_i s\kappa_i c\phi_i + s\zeta_i s\phi_i) s\theta_i + c\zeta_i c\kappa_i c\theta_i \right] s\chi_i \\ (c\zeta_i s\kappa_i s\phi_i - s\zeta_i c\phi_i) s\chi_i + \left[-(c\zeta_i s\kappa_i c\phi_i + s\zeta_i s\phi_i) s\theta_i + c\zeta_i c\kappa_i c\theta_i \right] c\chi_i \end{bmatrix}. \tag{2.3}$$

and after a rotation of the vector (2.3) by the angle σ_i around the axis $Q_i Z_i$ of this system, we obtain

$$U_i^{\sigma} = \begin{bmatrix} c\sigma_i & -s\sigma_i & 0 \\ s\sigma_i & c\sigma_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{ix} \\ U_{iy} \\ U_{iz} \end{bmatrix} = \begin{bmatrix} U_{ix} c\sigma_i - U_{iy} s\sigma_i \\ U_{ix} s\sigma_i - U_{iy} c\sigma_i \\ U_{iz} \end{bmatrix}. \tag{2.4}$$

As presented in the study (Bil, 2012), every torus has four circular sections that pass through one point (two that meet the condition $r_i > s_i$ and two $r_i < s_i$) that differ with the signs of the angles χ_i and θ_i of the position of the axes of the circles with the radius s_i in relation to the main axis of the torus; cf. Fig.1. The position of these circles can be defined through the vectors of the position of their centres R_i ($i=1$, 2) in relation to the system $Q_i X_i Y_i Z_i$ and the direction versors of the axes Z_{11} and Z_{21} respectively

$$R_i = \begin{bmatrix} c\sigma_i & -s\sigma_i & 0 \\ s\sigma_i & c\sigma_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\sigma_i \\ s\sigma_i \\ 0 \end{bmatrix}, \tag{2.5}$$

$$Z_i = \begin{bmatrix} c\theta_i c\phi_i c\sigma_i + \left(c\chi_i s\phi_i - s\chi_i s\theta_i c\phi_i\right) s\sigma_i \\ c\theta_i c\phi_i s\sigma_i - \left(c\chi_\mathrm{i} s\phi_\mathrm{i} - s\chi_i s\theta_i c\phi_i\right) c\sigma_i \\ -s\chi_i s\phi_i - c\chi_i s\theta_i c\phi_i \end{bmatrix} \tag{2.6}$$

where: $i=1, 2, 3, 4$ is the number of the torus and the system of coordinates related to the torus (Fig.2).

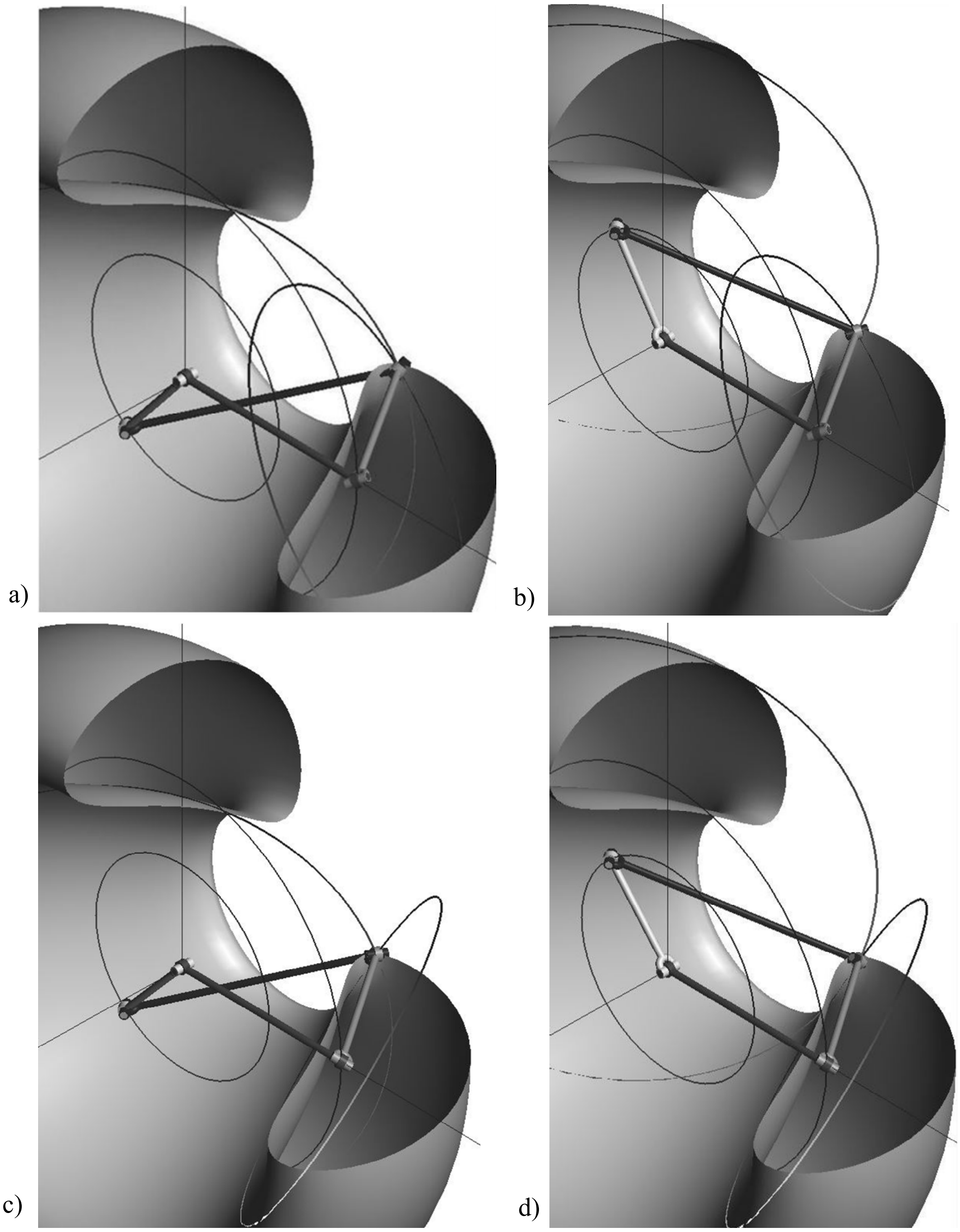

Fig.2. Four variants (assembly diagrams) of universal four-bar 4R mechanisms that are based on identical tori.

3. Spatial five-bar 5r linkage

A five-bar 5R mechanism can be formed from a combination of two four-bar 4R linkage through their series connection, so that the output lever of the first of the mechanisms to be combined coincides with the input lever of the other one (Fig.2). This corresponds to the situation of an intersection of two tori in such a manner that the circle on which the point moves on the surface of one torus could coincide with the circle on which the point moves on the surface of the other torus. After the combination, coinciding lever need to be removed.

For this to be realized, the following conditions need to be fulfilled:

- the centres of circles with the same radiuses coincide;
- the axes of the revolute kinematic pairs that are located on the coinciding ends of the links need to coincide with compliant or opposite versors.

This leads to a simultaneous satisfaction of the following equations

$$R_1=Q_3 \quad \text{and} \quad s_1=r_3 \quad \text{and} \quad Z_{11}=\pm Z_3 \quad \text{and} \quad U_1=\pm Z_{31}, \tag{3.1}$$

or

$$R_1=Q_4 \quad \text{and} \quad s_1=r_4 \quad \text{and} \quad Z_{11}=\pm Z_4 \quad \text{and} \quad U_1=\pm Z_{41}, \tag{3.2}$$

or

$$R_2=Q_3 \quad \text{and} \quad s_2=r_3 \quad \text{and} \quad Z_{21}=\pm Z_3 \quad \text{and} \quad U_2=\pm Z_{31}, \tag{3.3}$$

or

$$R_2=Q_4 \quad \text{and} \quad s_2=r_4 \quad \text{and} \quad Z_{21}=\pm Z_4 \quad \text{and} \quad U_2=\pm Z_{41}. \tag{3.4}$$

If we remove from the arrangement of the mechanisms formed in this way those links that coincide, then in each of the 16 variants a five-bar with single degree-of-freedom will be formed with revolute pairs that realizes a conversion of the complete rotation of the input links to a complete rotation of the output link between warped axes.

We shall consider one of the possible solutions on the basis of Eqs (3.2), when the circle with the radius s_1 with the centre in the point R_1 of the first torus coincides with the circle with the radius r_4 with the centre in the point Q_4 of the fourth one. The versor of the axis Z_4 of the fourth torus and the versor of the axis Z_{11} of the first torus will be identical (or they may be opposite). At the same time, the directional versors of the revolute pairs $U_1=Z_{41}$ must coincide at the points S_1 and R_4 (or be opposite to one another: $U_1=-Z_{41}$).

In order to write the arrangement of the five-bar mechanism in the form of equations, torus no. 1 and torus no. 4 (Fig.3) need to be written in the common immovable system of coordinates, e.g., $Q_1X_1Y_1Z_1$. As a result of this, the system $Q_4X_4Y_4Z_4$ will be shifted along the OX axis to the distance r_1 and rotated along this axis by the angle χ_1 and rotated around the axis Q_4Y_4 by the angle θ_1. The coincidence of a radius with the length $s_1 = r_4$ requires for the rotation angle of these radiuses to be the same, i.e., $\varphi_1=\sigma_4$.

In connection with this, the vectors that describe the position of the vector of the point S_4 of torus no.4, on the basis of Eq.(2.2) with the index $i=4$, in immovable system of coordinates no.1 will be as follows

$$S_4^1 = \begin{bmatrix} c\sigma_1 & -s\sigma_1 & 0 \\ s\sigma_1 & c\sigma_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi_1 & -s\chi_1 \\ 0 & s\chi_i & c\chi_1 \end{bmatrix} \begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ 0 & 1 & 0 \\ -s\theta_1 & 0 & c\theta_1 \end{bmatrix} \begin{bmatrix} S_{4x} \\ S_{4y} \\ S_{4z} \end{bmatrix} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}, \tag{3.5}$$

and after conversions

$$S_4^1 = \begin{bmatrix} \left(S_{4x}c\theta_1 + S_{4z}s\theta_1\right)c\sigma_1 - \left[S_{4y}c\chi_1 - \left(-S_{4x}s\theta_1 + S_{4z}c\theta_1\right)s\chi_1\right]s\sigma_1 + r_1 \\ \left(S_{4x}c\theta_1 + S_{4z}s\theta_1\right)s\sigma_1 + \left[S_{4y}c\chi_1 - \left(-S_{4x}s\theta_1 + S_{4z}c\theta_1\right)s\chi_1\right]c\sigma_1 \\ S_{4y}s\chi_1 + \left(-S_{4x}s\theta_1 + S_{4z}c\theta_1\right)c\chi_1 \end{bmatrix}. \tag{3.6}$$

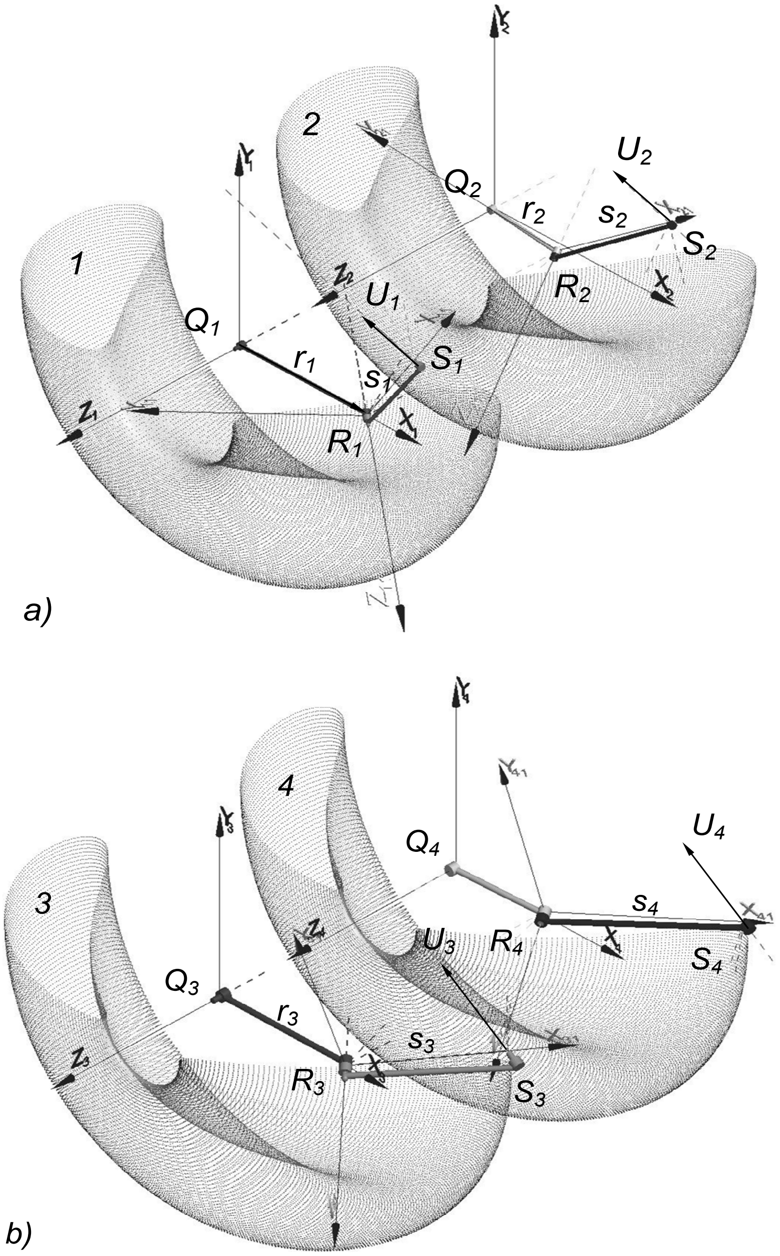

Fig.3. System of four tori that gives grounds for the creation of five-bar 5R linkage.

The direction versor U_{41} of the axis of the revolute pair at the point S_4 in immovable system of coordinates no.1, using Eq.(2.4) with the index $i=4$, will be presented as follows

$$U_{41}^{1}=\begin{bmatrix} c\sigma_1 & -s\sigma_1 & 0 \\ s\sigma_1 & c\sigma_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi_1 & -s\chi_1 \\ 0 & s\chi_{\mathrm{i}} & c\chi_1 \end{bmatrix}\begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ 0 & 1 & 0 \\ -s\theta_1 & 0 & c\theta_1 \end{bmatrix}\left[U_{41}\right], \tag{3.7}$$

and after conversions

$$U_{41}^{1}=\begin{bmatrix} \left(U_{41x}c\theta_1+U_{41z}s\theta_1\right)c\sigma_1-\left[U_{41y}c\chi_1-\left(-U_{41x}s\theta_1+U_{41z}c\theta_1\right)s\chi_1\right]s\sigma_1 \\ \left(U_{41x}c\theta_1+U_{41z}s\theta_1\right)s\sigma_1+\left[U_{41y}c\chi_1-\left(-U_{41x}s\theta_1+U_{41z}c\theta_1\right)s\chi_1\right]c\sigma_1 \\ U_{41y}s\chi_1+\left(-U_{41x}s\theta_1+U_{41z}c\theta_1\right)c\chi_1 \end{bmatrix}. \tag{3.8}$$

Because in each of the variants (7 – 10), torus no.1 coincides with torus no.2, and torus no.3 with torus no.4, then by analogy to torus 4, the following will be presented: the point S_{31} and the versor U_{31} that describe the positions of the collaborating elements of the mechanism related to torus no.3 (Fig.4).

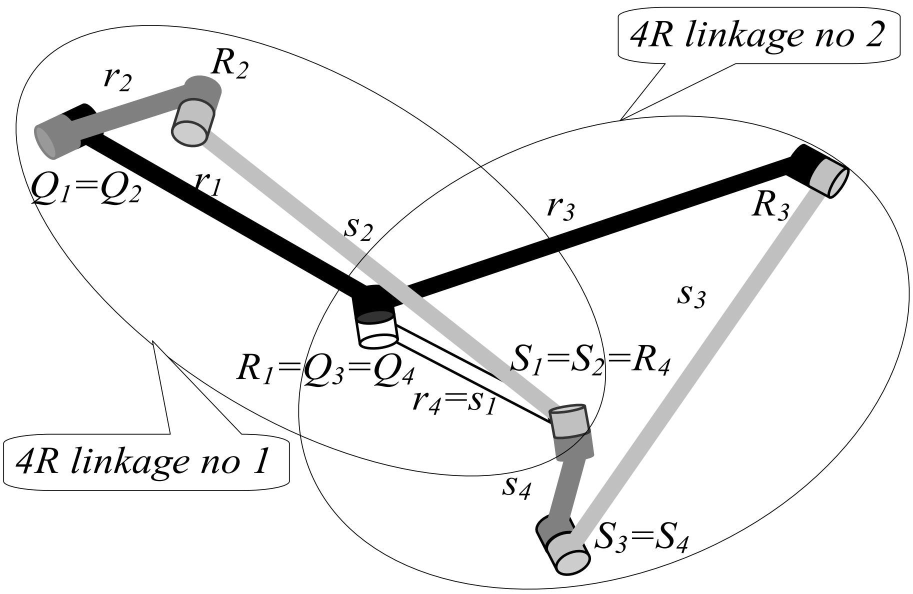

Fig.4. Five-bar 5R spatial linkage resulting from the connection of two four-bar 4R linkage.

Using Eqs (3.1) or (3.3) or (3.4), connections can be obtained that will be described with those formulae that are analogical to Eqs (3.5)-(3.8).

4. Conclusions

The present paper contains a geometric description of a five-bar 5R mechanism in a general form with the use of tori geometry in a universal form. A special case of this mechanism is the mechanism obtained by Goldberg (1943) out of a combination of two G.T. Bennett mechanisms (Bennett, 1903; Baker, 1979).

The model presented permits an analysis and synthesis of the mechanism with any mutual positioning of the axis of rotation. The five-bar mechanism allows a conversion of the rotational motion into rotational motion in the range of the round angle between the warped axes at any angle.

Due to the complexity of the model, the solution of those equations by means of which all the required linear and angular dimensions can be calculated is possible only with approximate methods. For this purpose, the authors used optimization algorithms. Independently of this, a Davidon–Fletcher–Powell algorithm was used that was realized in the Delphi 10 system and a Generalized Reduced Gradient algorithm that was used in the Solver applet in the MS EXCEL software (Microsoft Excel Solver User's Guide).

Nomenclature

$i=1, 2, 3, 4$	– the number of the torus and its dimensions
$Q_i X_i Y_i Z_i$	– immovable system of coordinates
R_i, S_i	– the points and the vectors of the positions in the absolute system of coordinates *OXYZ*
$R_i X_{i1} Y_{i1} Z_{i1}$	– auxiliary system of coordinates
$\boldsymbol{r}_i, \boldsymbol{s}_i$	– the radiuses and the vectors that determine the position of the nodal points R_i, S_i
$\phi_i, \sigma_i, \chi_i, \theta_i, \kappa_i, \zeta_i$	– the angle that determines the position of the nodal points R_i, S_i and versors U_i

References

Baker J.E. (1979): *The Bennett, Goldberg and Myard linkages – in perspective*. – Mech. and Mach. Theory 14 (4), pp.239-253.

Bennett G.T. (1903): *A new mechanism*. – Engineering, London, vol.76, pp.777–778.

Bil T. (2010): *Bennett linkage in the geometry of tori*. – Acta Mechanica et Automatica, vol.4, No.1, pp.5-8.

Bil T. (2011): *Kinematic analysis of a universal spatial mechanism containing a higher pair based on tori*. – Mechanism and Machine Theory, vol.46, pp.412-424.

Bil T. (2012): *Analysis of the Bennett linkage in the geometry of tori*. – Mechanism and Machine Theory, vol.53, pp.122-127.

Bil T. (2012): *The new general model four-bar spatial linkage*. – Int. J. of Applied Mechanics and Engineering, vol.17, No.3, pp.677-688.

Budniak Z. and Bil T. (2012): *Simulation of the movement of four-bar spatial linkage*. – Int. J. of Applied Mechanics and Engineering, vol.17, No.3, pp.723-732.

Goldberg M. (1943): *New five-bar and six-bar linkages in three dimensions*. – ASME J. Mechanisms 65, pp.649-661.

"Microsoft Excel Solver User's Guide" for Windows, version 3.0, pp. 2.

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