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# CRITERIA FOR THE MINIMUM OPERATION LENGTH OF INTERNAL FORCES AS A FUNCTION OF THE DEVELOPMENT OF AN OPTIMUM STRUCTURE OF MACHINERY STRUCTURAL COMPONENTS

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The present study analyzes the operation length of internal forces (DDSW) understood as the length of the flow of internal forces along the shortest possible internal routes. The operation length of internal forces is determined on the basis of stresses and the given volume in the constructional space. The minimum DDSW of the structure satisfies the criterial conditions of the most rigid structure, where the potential energy of deformation and the deformation energy potential is the same in the whole volume and thus the potential gradient is zero.

**Key words:** operation length of internal forces, topological optimization, optimization criteria

## 1. Introduction

The formation of structural components and structures is applicable to computer-aided design. A bearing structure can be formed from a given volume of material in an infinite number of ways. Based on the conditions for the criteria of the formation of the optimum and the most rigid structure, machinery or a structural component should be shaped so that the transfer of external active loads (generalized forces of external and technological impacts) and passive loads (reactions of bonds) should take place on the smallest operation length of internal forces (DDSW) of the bearing structure and so that the potential of elastic strain energy at each point of the structure is the same [1]. Traditional optimization is focused on the alignment of stresses [2] and [3].

The paper covers an optimal formation of the bearing structure in the two-dimensional (2D) and three dimensional (3D) space of construction. The operation length of internal forces is examined (DDSW) understood as the length of the flow of internal forces along the shortest possible internal routes. The operation length of internal forces is determined on the basis of stress σ and a given volume $V_0$ in the area of construction. As a quantitative criterion of the operation length of internal forces $D$, we accept the integral of stress σ referred to volume $V_0$ = const. of the whole structure [2] and [4]

$$D = \int_{V_0} dV \tag{1.1}$$

where, in the case of base loads (tension, compression, shear, bending, torsion) the function σ($x$, $y$, $z$) is a function of normal or tangential stresses, while for a complex state of stresses and deformations, it represents reduced stresses determined on the basis of one of strength hypotheses.

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For the case of base loads, owing to the known stress distribution function, it is possible to precisely define the integral (1.1) [2], while for two- and three-dimensional systems, we usually do not know this function, hence a discrete computational model is then used, which is adequate to currently widely used methods of calculation.

Since the late sixties of the past century, an optimization of the topology of a body with a continuous and not merely rod structure has been used. As examples of some of the earliest studies on continuous structures, one may cite an article from of [5], which considered theoretical issues of an optimal structure design in order to obtain a structure of a maximum rigidity. In [6] the optimum thickness was determined of a plate construction. It was not until the late eighties that studies occurred that provided grounds for further research [3, 4]. In the nineties and later, until now, there has been a rapid development of topology optimization of continuous bodies, both homogeneous and composite ones [7].

## 2. Operation length internal forces in two-dimensional discrete design space

In [8] the task of an optimization and the operation length of internal forces in one- and two-dimensional space taking into account the continuity of the construction space was examined. In this paper, we will consider the task of determining the minimum DDSW in the most rigid structure in a given discrete construction space. The optimization task will be divided into two stages of optimum formation. The first stage will include the optimum distribution of material properties in a predetermined volume. In the second stage, we will convert the material properties into the real properties of the construction materials used and dimensioning will be performed as well.

As an example, we will consider a two-dimensional design space $\Omega(x, y, h)$ of a disc type, where $h$ is the thickness of the structure (Fig.1).

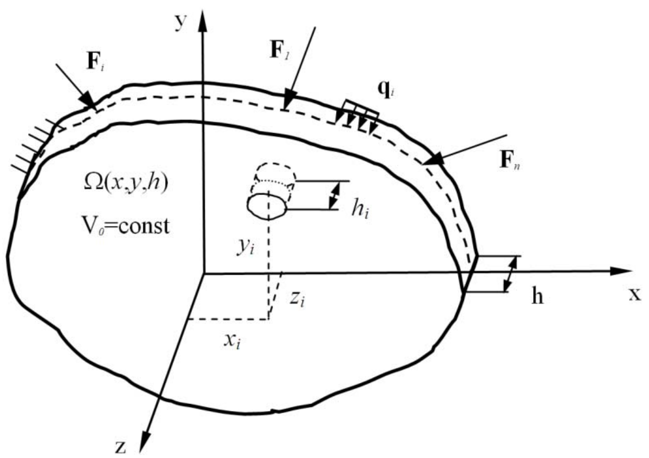



Fig.1. Discrete design space

When performing discretization of space $\Omega(x, y, h)$ into $n$ elements, the total deformation energy $U$, volumetric strain energy $U_V$ and shear strain energy $U_p$ and constraint $\Theta_E$ for volume $V_0$ and Young modulus $E_V$ in each $r$-th iteration take the following form

$$U = \sum_{i=1}^{n} \left[ \frac{1}{2E_i} \left( \sigma_{x_i}^2 - 2\mu_i \sigma_{x_i} \sigma_{y_i} + \sigma_{y_i}^2 \right) + \frac{1+\mu_i}{E_i} \tau_{x_i y_i}^2 \right] V_{0i} , \tag{2.1}$$

$$U_V = \sum_{i=1}^{n} \left[ \frac{1-2\mu_i}{6E_i} \left( \sigma_{x_i} + \sigma_{y_i} \right)^2 \right] V_{0i} , \tag{2.2}$$

$$U_p = \sum_{i=1}^{n} \left[ \frac{1+\mu_i}{3E_i} \left( \sigma_{x_i}^2 - \sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 \right) + \frac{1+\mu_i}{E_i} \tau_{x_i y_i}^2 \right] V_{0i} \,, \tag{2.3}$$

$$\Theta_E = \sum_{i=1}^{n} E_{0ji} V_{0i} - E_V = 0 \,, \quad E_V = \sum_{i=1}^{n} E_i V_{0i} = \text{const} \,, \quad V_0 = \sum_{i=1}^{n} V_{0i} = \text{const} \tag{2.4}$$

where: $V_{0i} = s_i h_{0i}$ it is the volume of $i$-th element of surface $s_i$ and initial thickness $h_{0i}$, $\sigma_{x_i}$, $\sigma_{y_i}$, $\tau_{x_i y_i}$ - stress in the $i$-th element, $\mu_i$ – Poisson's ratio of the $i$-th element, $E_{0ji}$ - the initial value of the $j$-th material (Young modulus) in the $i$-th element.

Having regard to Formulas (2.1), (2.2), (2.3) and Constraint (2.4) and accepting the Lagrangian functional, we obtain a dependence to determine the Young's moduli for the first optimization step

$$E_{ir} = \frac{\sum_{i=1}^{n} \left( E_i s_i h_{0i} \right)_{r-1} \sqrt{\left( \sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 + 2\left(1+\mu_{0ji}\right)\tau_{x_i y_i}^2 \right)_{r-1}}}{\sum_{i=1}^{n} \left[ \left( s_i h_{0i} \right)_{r-1} \sqrt{\left( \sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 + 2\left(1+\mu_{0ji}\right)\tau_{x_i y_i}^2 \right)_{r-1}} \right]} \tag{2.5}$$

where: $E_{ir}$ the sought Young's modulus in the $r$-th iteration, $\mu_{0ji}$ - Poisson's ratio of the given $j$-th type of construction material.

The calculations are stopped when the difference $E_{ir} - E_{i(r-1)} \le \varepsilon$ ($\varepsilon$–low set point), at each point of the construction space is met.

For further calculations, we accept the function $f(V,E) = U_r - U_w$ of the difference in the potential energy of deformation $U_r$ in $r$-th iteration and the energy of deformation for the construction of the sought optimal dimensions $U_w$. This dependency will be determined as follows

$$\begin{aligned} f(V,E) = &\sum_{i=1}^{n} \left[ \frac{1}{2E_{ir}} \left( \sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 \right)_r + \frac{1+\mu_{0ji}}{E_{ir}} \left( \tau_{x_i y_i}^2 \right)_r \right] V_{0i} + \\ &- \sum_{i=1}^{n} \left[ \frac{1}{2E_{0ji}} \left( \sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 \right)_r + \frac{1+\mu_{0ji}}{E_{0ji}} \left( \tau_{x_i y_i}^2 \right)_r \right] V_i \end{aligned} \tag{2.6}$$

where: $V_i = s_i h_i$ is the volume of the separated $i$-th volume with area $s_i$ and thickness $h_i$ sought. The constraints will be written as

$$\Theta_w = \sum_{i=1}^{n} E_{ir} V_{0i} - E_V = 0 \,, \quad V_0 = \sum_{i=1}^{n} V_i = \text{const} \,, \quad E_V = \sum_{i=1}^{n} E_{0ji} V_i = \text{const} \,. \tag{2.7}$$

Accepting the flux of forces $R_i = \sqrt{\left[ \sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 + 2\left(1+\mu_{0ji}\right)\tau_{x_i y_i}^2 \right]_{r-1}}\, h_i = \text{const}$ for each iteration and taking into account (2.6) and (2.7), we obtain a dependence to determine the converted thickness of the modules Young $E_{ir}$ set in the first optimization step

$$h_i = \frac{\sum_{i=1}^{n} \mathrm{E}_{ir}\, s_i h_{0i} \sqrt{\sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 + 2\left(1+\mu_{0ji}\right)\tau_{x_i y_i}^2}}{\sum_{i=1}^{n} \mathrm{E}_{0ji} s_i \left[\sqrt{\left(\sigma_{x_i}^2 - 2\mu_{0ji}\sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 + 2\left(1+\mu_{0\,ji}\right)\tau_{x_i y_i}^2\right)}\right]}. \tag{2.8}$$

Dependence (2.8) to determine the converted thickness of the modules Young minimizes the value of the function (2.6).

Using this methodology, structure calculations were performed in two construction spaces shown in Figs 2a and 2b. The finite element method was used in the calculations. The construction was discretized and forces: $F_1=800\ N$, $F_2=760\ N$, $F_3=550\ N$, $F_4=200\ N$ were applied. The initial (starting) thickness of elements was chosen so that the volume $V_0 = 1.3682\ 10^{-4}\ m3$ could be the same for each structure. The grid nodes and elements, and load and fixing points are shown in Figs 2a, 2b.

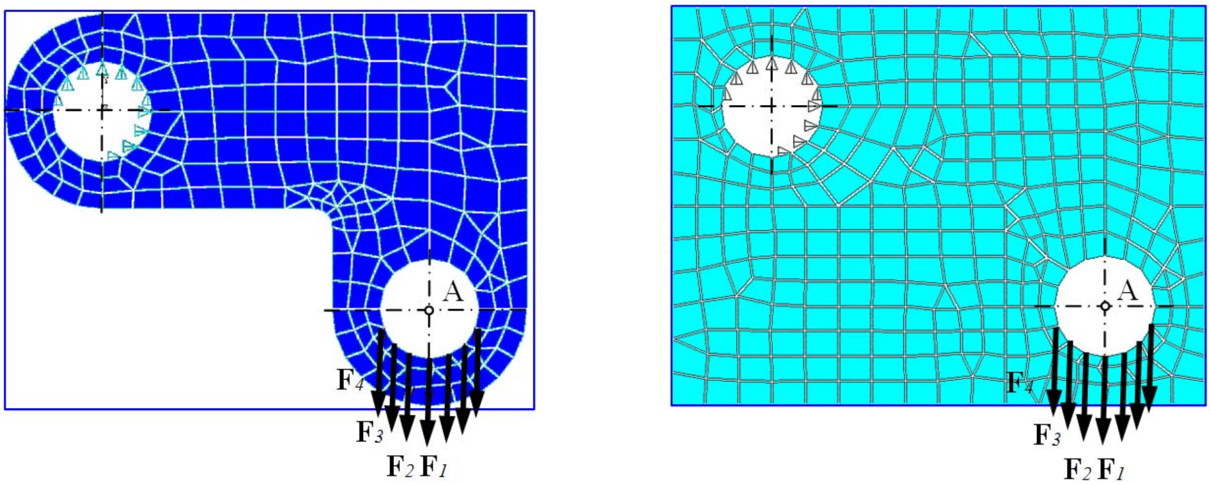



Fig.2. Spaces of seeking design with minimal DDSW: a) angle bracket, b) rectangle.

In the discretized area in a complex state of stress, DDSW will be determined from the following relation

$$D = \sum_{i=1}^{n} \sigma_i\ \mathrm{V}_i \tag{2.9}$$

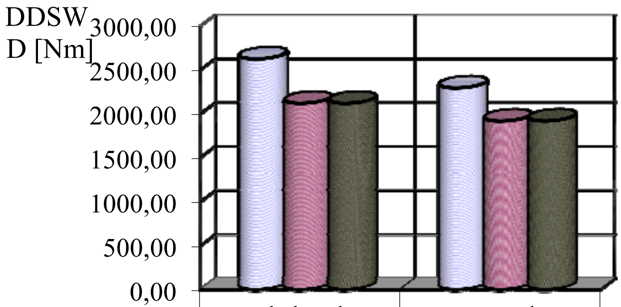
where: $\sigma_i$ is stress reduced according to H-M-H (Huber–Mises- Hencky) hypothesis, $V_i$ is the volume of $i$-th finite element.

For a two-dimensional stress state in $x_i$, $y_i$, the coordinates HMH stresses in the middle of the finite element and in the volume finite elements are equal to

$$\sigma_i = \sqrt{\sigma_{x_i}^2 - \sigma_{x_i}\sigma_{y_i} + \sigma_{y_i}^2 + 3\tau_{x_i y_i}^2}\,, \qquad V_i = s_i h_i\,. \tag{2.10}$$

After the calculations, it was found that the unit value of the potential energy of deformation in finite elements differed by a small value, which means that the gradient of the rate of change was not large. This is consistent with the definition of the most rigid structure, where the unit potential energy of deformation (potential) is the same at each point of design.

On the basis of the calculations, graphs were drawn of DDSW change in the design space (Fig.3). Figure 3 presents how DDSW changed in zero and final iteration as well as during the conversion of a certain type of material $E_i$ in components into the component's thickness $h_i$.




| | angle bracket | rectangle |
|---|---|---|
| iter. 0 | 2622,91 | 2292,46 |
| iter. 15 | 2117,67 | 1915,11 |
| iter. E/h | 2117,54 | 1916,01 |

Fig.3. The total length of the forces of internal iterations

On the basis of the change in the operation length of internal forces $D_i=\sigma_i V_i$ in finite elements and the total change in DDSW elements, it can be noted that with increasing the stiffness of the elements and a compensation of unit deformation energy gradients in the construction space, the length of their inner forces decreases. The DDSW value in elements with the accepted elasticity $E_V=\sum E_{0ji}V_i$=const., which determines the distribution of complex materials in the construction is the same as the converted thicknesses of elements. It is evident that the structure with the volume $V_0$=const. and the accepted elasticity $E_V$=const. that has the potential that is equal at each point in the construction space, has the lowest DDSW equal to $D = \int_{V_0} dV = \min$ .

## 3. Operation length of internal forces in three dimensional discrete design space

We will consider a discretized three-dimensional design space Ω (*x*, *y*, *z*), wherein a portion of a structure contained in that space is not used due to its low load. In this case, we can improve the shape using the method of topology optimization. Removing the limits of a predetermined percentage reduction in the volume $V_0$ less burdened discreet volume $V_i$, we obtain a new design of the construction of a smaller volume.

When solving the problem of topology optimization, the following assumptions were adopted [9]:

- Continuum medium will be tested, as a homogeneous and isotropic medium;
- The material is linear-elastic ($E$ = const.);
- It is assumed that the strain tensor is linear (analysis for small strains);
- During the optimization process, the problem is considered for the permanent constructional space Ω (*x*, *y*, *z*) with dimensions of $0.25\times0.25\times1\ m$;
- An approach is used of the finite element method, the updated Lagrangian description and the calculus of variations;
- In the optimization process, a certain initial volume (*w*) $V_0$ is possessed of the body reduced by *25*%;
- The adopted optimization criterion is a maximized stiffness function of the component with restrictions imposed on the volume [10, 11, 12];

- The change is examined in the operation length of internal forces (DDSW) in the output and the optimized construction with regard to the abovementioned criteria [13, 14, 15, 16].

On the example of the bearing structure (Fig.4a), a change was examined in DDSW in the given discrete construction space Ω ($x$, $y$, $z$) and a change to the unit and total potential energy of deformation (the gradient of the unit potential deformation energy) in the initial and optimum structure.

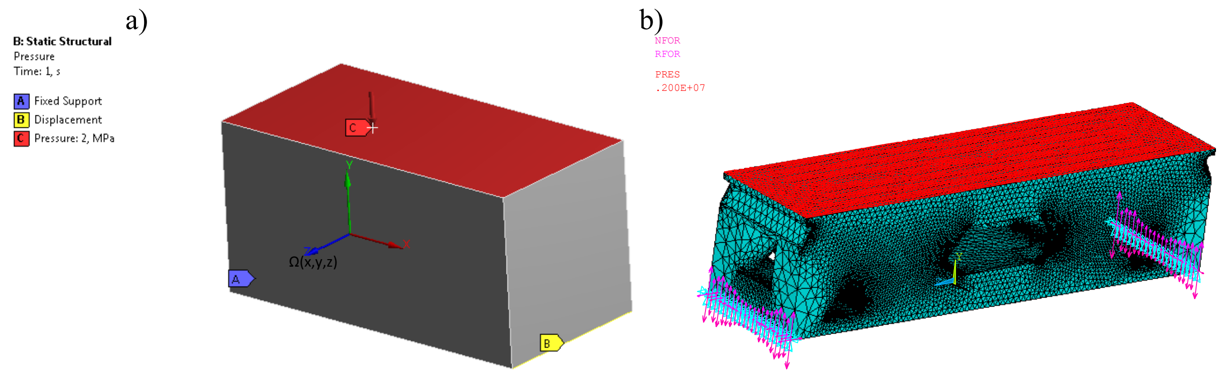



Fig.4. Load, geometry and imposed constraints on the bracket: a) design input $V=0.0625\ m^3$; b) optimal design $V=0.0469\ m^3$.

The bearing structure was loaded on the upper surface with a pressure of *2* MPa and constraints were imposed on it on A and B edges (Figs 4a, 4b). By limiting the size of the volume reduction to *25*% of the bearing structure and using topology optimization methods, a three-dimensional structural element was obtained with the shape as shown in Fig.4b. It was calculated using the ANSYS software and the SOLID 185-3-D 8-Node Structural Solid finite element. To verify the influence of the grid calculations, the output element was divided into 32000 and 500000 and finite elements. The operation length of internal forces was determined according to Formula (2.9). The calculation results are presented in Tab.1.

Table 1. Characteristic data prior to and after optimization of the bracket.

| | Before optimization | Before optimization | After optimization |
|---|---|---|---|
| The number of finite elements | *32000* | *500000* | *439149* |
| Volume [*m3*] | *0.06249* | *0.06249* | *0.04593* |
| DDSW [*Nm*] | *661486.97* | *663588.08* | *655599.57* |
| The potential energy of deformation [*J*] | *25.397* | *27.573* | *28.843* |
| The unit energy of deformation [*J/m3*] min<br>max | *5.343*<br>*42816.036* | *2.570*<br>*276744.000* | *0.312*<br>*91369.106* |
| Equivalent stress (H-M-H) [MPa] min<br>max | *1.121*<br>*115.080* | *0.901*<br>*292.450* | *0.316*<br>*198.810* |

Based on the data obtained from the numerical experiment, it may be observed that the grid has a significant impact on the optimization results. After a division into 32000 components; the results were inadequate, wherein the grid has a smaller impact on DDSW (Tab.1). When divided into a similar number of finite elements in the output and optimal construction, the results are consistent with expectations. Considering a reduction of the structural element by *25*%, it can be concluded that Huber-Mises stresses (Figs 5a, 5b) decreased in the optimal structure. The unit potential deformation energy and became more even in the volume , which means a greater alignment of the unit potential energy gradient. The energy potential is more quantitatively even and it is smaller than in the input structure (Figs 6a, 6b), while the conventional rigidity expressed by the total deformation energy increased by *4.6*%, which is in line with expectations.

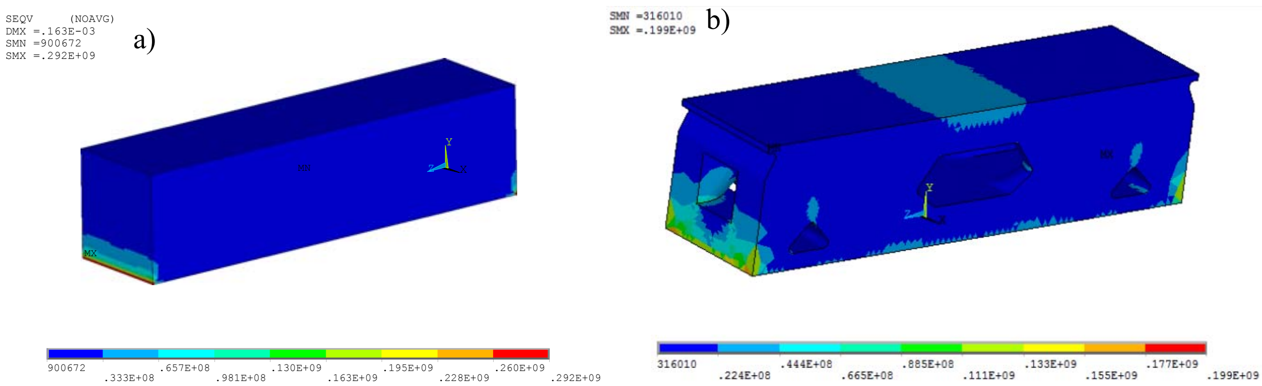



Fig.5. Distribution of equivalent stress (H-M-H): a) design input; b) optimal design.

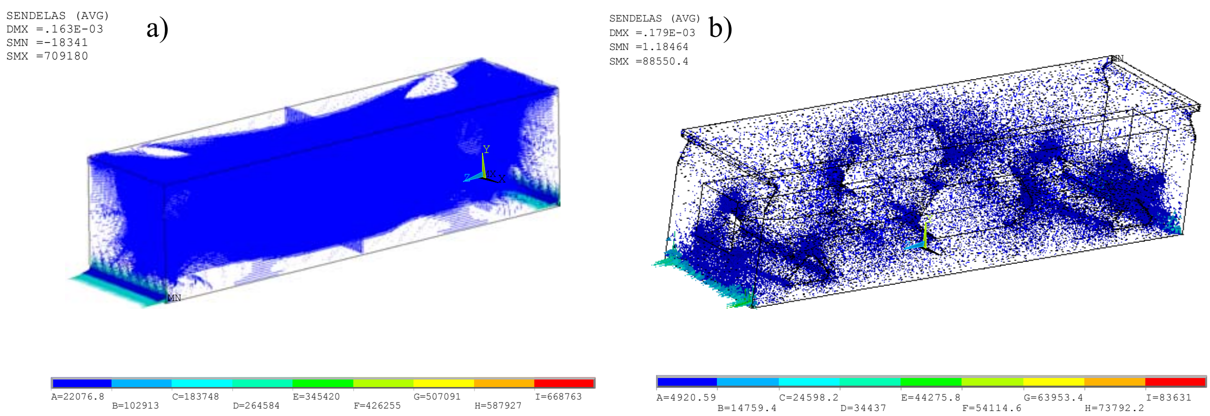



Fig.6. Map of elastic strain energy density: a) design input; b) optimal design.

The duration of the internal forces (DDSW) decreased by *1.2*%, thus reflecting the reduction of the respiratory flow of internal forces despite a reduction in the volume of construction of the optimal *25*%. For the decomposition of a material with the same volume of space in the construction, the construction of optimal DDSW was greatly reduced [8].

## 4. Summary

By identifying DDSW in the linear structure, we mentally equate it to a rod that is loaded along the axis with geometric and physical linearity [2]. This rod always meets the criterial conditions of the most rigid element, where DDSW, the potential energy of deformation, the deformation energy potential are the same across the whole volume, and thus the potential gradient is zero. The results obtained indicate the possibility of the use as a function of the operation length of internal forces with a limitation as regards the volume of the structure in given area of the construction.

## Nomenclature

$D$ – the length of the activity of the internal forces criterion length of operation of internal forces
$E_{0ji}$ – initial value of the *j*-th material (Young's Modulus) in the *i*-th element
$E_{ir}$ – Young's Modulus in the *r*-th iteration
$h_i$ – converted thickness of the Young $E_{ir}$
$R_i$ – assuming stream forces
$U$ – total deformation energy
$U_V$ – volumetric strain energy
$U_p$ – strain energy
$V_i$ – volume of the separated *i*-th volume about area $s_i$ and sought thickness $h_i$
$V_{0i}$ – volume of the *i*-th element of the surface $s_i$ and the initial thickness $h_{0i}$
$\varepsilon$ – low set point
$\Theta_E$ – limiting for volume $V_0$ and Young's Modulus $E_v$
$\mu_i$ – Poisson's ratio of the *i*-th element
$\mu_{0ji}$ - Poisson's ratio *j*-th the type of construction material
$\sigma(x, y, z)$ – function of normal and tangential stress, while the complex state of stress and deformation is representative of the equivalent stresses determined on the basis of one of the hypotheses strength
$\sigma_{xi}$, $\sigma_{yi}$, $\tau_{xi\,yi}$ – stress in the *i*-th element
$\Omega(x, y, h)$ – dimensional design space

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