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Brief note

EDGE WAVES OVER A SHELF

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The problem considered in this paper is the derivation of properties of edge waves travelling along a submerged horizontal shelf. The problem is formulated within the framework of the linearized theory of water waves and Havelock expansions of water wave potentials are used in the mathematical analysis to obtain the dispersion relation for edge waves in terms of an integral. Appropriate multi-term Galerkin approximations involving ultra spherical Gegenbauer polynomials are utilized to obtain a very accurate numerical estimate for the integral and hence to derive the properties of edge waves over a shelf. The numerical results are illustrated in a table and curves are presented showing the variation of frequency of the edge waves with the width of the shelf.

Key words: shelf, edge waves, Havelock expansion, Galerkin approximation, Gegenbauer polynomial, dispersion relation.

1. Introduction

Edge wave solutions to the linearized theory of water waves are well known in the literature for a variety of bottom topographies. The only explicit solution exists for edge waves over a uniform sloping beach (cf. Stokes [1], Ursell [2], Jones [3], Roseau [4], Grimshaw [5]). The shallow water dispersion relation for edge wave modes over a shelf was extensively studied by Snodgrass *et al*. [6], Summerfield [7] and Longuet-Higgins [8] and it can be shown that for a fixed geometry, the number of modes increases indefinitely with the increase of wave numbers. The full linearized theory was utilized by Evans and McIver [9] to derive the properties of edge waves over a shelf.

In this paper we derive the properties of edge waves travelling along a submerged horizontal shelf bounded on one side by a vertical wall extending through the free surface and on the other by a vertical drop from the shelf to a deeper region of constant water depth extending horizontally indefinitely. The problem is formulated within the framework of the linearized theory of water waves and Havelock expansions of water wave potentials are used in the mathematical analysis to obtain the dispersion relation for edge waves in terms of an integral. Appropriate multi-term Galerkin approximations involving ultra spherical Gegenbauer polynomials are utilized to obtain a very accurate numerical estimate for the integral and hence to derive the

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properties of edge waves over a shelf. The numerical results are illustrated in a table and curves are presented showing the variation of frequency of the edge waves with the width of the shelf.

2. Formulation of the problem

We consider the motion in an inviscid, homogeneous, incompressible liquid which is supposed confined in the horizontal shelf. Cartesian axes are chosen with the mean free surface the (x, z) plane, z being directed along the straight coastline and y vertically downwards. The shallower water is of finite depth h_1 above the horizontal shelf of width a; the deeper water is of depth h_{21}. A simple sketch of the problem is given in Fig.1.

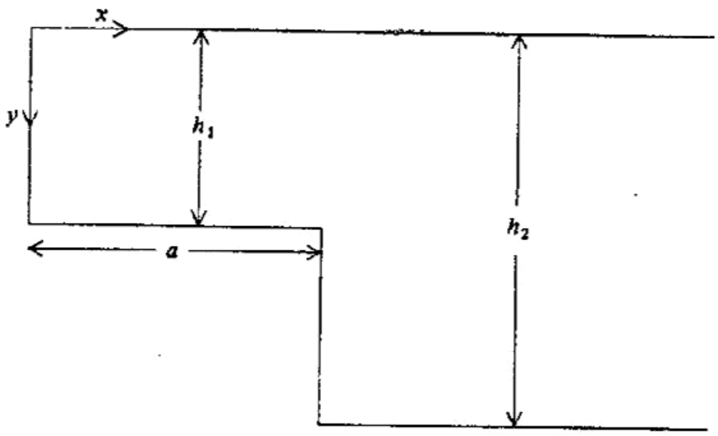

Fig.1. Geometry of the problem.

Assuming the linearized theory of water waves, the edge waves travelling along a submerged horizontal shelf can be described by the velocity potential $\mathrm{Re}\{\varphi(x,y)\exp(i\vartheta z - i\sigma t)\}$, where ϑ being the wave number in the z direction, σ being the frequency of the edge waves, then φ satisfies

$$\nabla^2\varphi - \vartheta^2\varphi = 0 \quad \text{in the fluid region}, \tag{2.1}$$

the free surface condition

$$K\varphi + \frac{\partial\varphi}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{2.2}$$

with $K = \sigma^2/g$ g being the gravity, the bottom conditions

$$\frac{\partial\varphi}{\partial y} = 0 \quad \text{on} \quad y = h_1, \quad 0 < x < a \quad \text{and} \quad y = h_2, \quad x > a, \tag{2.3}$$

the conditions on the vertical walls

$$\frac{\partial\varphi}{\partial x} = 0 \quad \text{on} \quad x = 0, \quad 0 < y \le h_1 \quad \text{and} \quad x = a, \quad h_1 < y \le h_2, \tag{2.4}$$

the edge condition

$$r^{1/3}\nabla\varphi \quad \text{is bounded as} \quad r \to 0, \qquad r = \sqrt{(x-a)^2 + (y-h_1)^2}\ , \tag{2.5}$$

r is the distance from the edge.

Our aim will be to find a dispersion relation between the wave frequency σ and the wave number ϑ such that non-trivial solutions to the above equations exist.

3. Method of solution

Since $\varphi_x(x,y)$ and $\varphi(x,y)$ are continuity across (a, 0) to (a, h_1), we can write

$$\left(\frac{\partial\varphi}{\partial x}\right)_{x=a+} = \left(\frac{\partial\varphi}{\partial x}\right)_{x=a-} = f(y), \quad \text{say,} \quad \text{for} \quad 0 < y < h_1, \tag{3.1}$$

$$(\varphi)_{x=a+} = (\varphi)_{x=a-} \quad \text{for} \quad 0 < y < h_1. \tag{3.2}$$

A solution for $\varphi(x,y)$ satisfying (2.1), (2.2), (2.3) can be represented as

$$\varphi(x,y) \to \begin{cases} -B_0 \dfrac{\cosh k_0(h_1-y)\cos t_0 x}{t_0 \sin t_0 a} + \sum\limits_{1}^{\infty} B_n \dfrac{\cos k_n(h_1-y)\cosh s_n x}{s_n \sinh s_n a}, & 0 < x < a, \\ \sum\limits_{0}^{\infty} A_n \dfrac{\cos\alpha_n(h_2-y)\exp(-p_n(x-a))}{p_n}, & x > a \end{cases} \tag{3.3}$$

where $t_0^2 = k_0^2 - \vartheta^2$, $s_n^2 = k_n^2 + \vartheta^2$, $p_n^2 = \alpha_n^2 + \vartheta^2$, $s_0 = it_0$, k_0 satisfies $k_0 \tanh k_0 h_1 = K$, k_n satisfies $k_n \tan k_n h_1 + K = 0$, α_n satisfies $\alpha_n \tan\alpha_n h_2 + K = 0$.

Using Eqs (3.3) in Eqs (3.1) and (3.2), we find

$$\begin{aligned} f(y) &= -\sum_{0}^{\infty} A_n \cos\alpha_n(h_2-y), \quad 0 < y < h_1, \\ &= B_0 \cosh k_0(h_1-y) + \sum_{1}^{\infty} B_n \cos k_n(h_1-y), \quad 0 < y < h_1, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \sum_{0}^{\infty} A_n \frac{\cos\alpha_n(h_2-y)}{p_n} &= -B_0 \frac{\cosh k_0(h_1-y)\cot t_0 a}{t_0} + \\ &+ \sum_{1}^{\infty} B_n \frac{\cos k_n(h_1-y)\coth s_n a}{s_n}, \quad 0 < y < h_1. \end{aligned} \tag{3.5}$$

Use of Havelock's [10] inversion theorem in Eq.(3.4) produces

$$A_n = \frac{-4\alpha_n}{2\alpha_n h_2 + \sin 2\alpha_n h_2} \int_0^{h_1} f(y) \cos \alpha_n (h_2 - y) dy \,, \tag{3.6}$$

$$B_0 = \frac{4k_0}{2k_0 h_1 + \sinh 2k_0 h_1} \int_0^{h_1} f(y) \cosh k_0 (h_1 - y) dy \,, \tag{3.7}$$

$$B_n = \frac{4k_n}{2k_n h_1 + \sin 2k_n h_1} \int_0^{h_1} f(y) \cos k_n (h_1 - y) dy \,. \tag{3.8}$$

Using Eqs (3.6), (3.7), (3.8) in Eq.(3.5), we find

$$\int_0^{h_1} F(u) M(y,u) du = \cosh k_0 (h_1 - y), \qquad 0 < y < h_1 \,, \tag{3.9}$$

$$\int_0^{h_1} F(y) \cosh k_0 (h_1 - y) dy = A \tag{3.10}$$

where

$$F(u) = \frac{4t_0}{B_0 \cot t_0 a} f(u) \,,$$

$$M(y,u) = \sum_0^{\infty} \frac{\alpha_n \cos \alpha_n (h_2 - y) \cos \alpha_n (h_2 - u)}{p_n (2\alpha_n h_2 + \sin 2\alpha_n h_2)} + \\ + \sum_1^{\infty} \frac{k_n \cos k_n (h_1 - y) \cos k_n (h_1 - u)}{s_n (2k_n h_1 + \sin 2k_n h_1)} \coth s_n a,$$

$$A = \frac{t_0 (2k_0 h_1 + \sinh 2k_0 h_1)}{k_0} \tan t_0 a \,. \tag{3.11}$$

It may be noted that the function $F(y)$ and the constant A are real. The integral Eq.(3.9) is to be solved by $(N+1)$ multi-term Galerkin approximations of $F(y)$ in terms of ultraspherical Gegenbauer polynomials $C_{2n}^{1/6}(y/h_1)$ by noting the behavior of $F(y) \sim (h_1 - y)^{-1/3}$ as $y \to h_1 - 0$ given by (cf. Dolai [11])

$$F(y) = \sum_{n=0}^{N} a_n f_n(y), \quad 0 < y < h_1 \tag{3.12}$$

where

$$f_n(y) = -\frac{d}{dy}\left[\exp(-Ky)\int_y^{h_1}\exp(Kt)\hat{f}_n(t)\,dt\right], \quad 0 < y < h_1,$$

with

$$\hat{f}_n(y) = \frac{2^{7/6}\Gamma(1/6)(2n)!}{\pi\Gamma(2n+1/3)h_1^{1/3}\left(h_1^2 - y^2\right)^{1/3}}C_{2n}^{1/6}(y/h_1).$$

The unknown coefficients a_n $(n = 0,1,2,\cdots,N)$ are obtained by solving the system of linear equations

$$\sum_{n=0}^{N} a_n \Re_{nm} = d_m, \qquad m = 0,1,2,\cdots,N \tag{3.13}$$

where

$$\Re_{nm} = 4(-1)^{n+m}\left[\sum_{r=1}^{\infty}\left\{\frac{k_r\cos^2 k_r h_1 \coth s_r a}{s_r(2k_r h_1 + \sin 2k_r h_1)}\frac{J_{2n+1/6}(k_r h_1)J_{2m+1/6}(k_r h_1)}{(k_r h_1)^{1/3}} + \right.\right.$$

$$\left. + \sum_{0}^{\infty}\frac{\alpha_r\cos^2\alpha_r h_2}{p_r(2\alpha_r h_2 + \sin 2\alpha_r h_2)}\frac{J_{2n+1/6}(\alpha_r h_2)J_{2m+1/6}(\alpha_r h_2)}{(\alpha_r h_2)^{1/3}}\right],$$

$$d_m = \frac{I_{2m+1/6}(k_0 h_1)}{(k_0 h_1)^{1/6}}\cosh k_0 h_1.$$

Once a_n $(n = 0,1,2,\cdots,N)$ are solved, the real constant A can be determined from Eq.(3.10)

$$A \approx \tilde{A} = \sum_{n=0}^{N} a_n d_n. \tag{3.14}$$

Thus, from Eqs (3.11) and (3.14), we find the dispersion relation

$$t_0 \tan t_0 a = \frac{k_0\tilde{A}}{2k_0 h_1 + \sinh 2k_0 h_1}. \tag{3.15}$$

The edge waves will exist if we can find solution of the dispersion relation (3.15).

4. Numerical results

For existence of edge waves, we solve the dispersion relation (3.15) numerically. To find the numerical solutions of the dispersion relation (3.15), we have to find the numerical estimate of $\tilde{A}$ in (3.14). Multi-term Galerkin approximations are used to obtain the numerical estimate for $\tilde{A}$. In the numerical computations, we take at most six terms to produce ac fairly accurate numerical estimate for $\tilde{A}$.

We display a representative set of numerical estimates for $\tilde{A}$ in Tab.1, taking N=*0*, *1*, *2*, *3*, *4* and *5* in the (N+*1*)- term Galerkin approximations and some particular values of the different parameters.

It is observed from Tab.1 that the computed results for $\tilde{A}$ converge very rapidly with N, and for $N \geq 3$ an accuracy of almost six decimal places is observed. It appears that the present method of numerical procedure for the numerical computations of $\tilde{A}$ is quite efficient.

Table 1. The computed results for $\tilde{A}$.

$Kh_2 = 0.2, \quad a/h_2 = 0.1$					
	$h_1/h_2 = 0.1$	$h_1/h_2 = 0.3$	$h_1/h_2 = 0.5$	$h_1/h_2 = 0.7$	$h_1/h_2 = 0.9$
N	$\tilde{A}$	$\tilde{A}$	$\tilde{A}$	$\tilde{A}$	$\tilde{A}$
0	0.641850	0.848978	0.833867	0.658619	0.358839
1	0.642881	0.852798	0.843405	0.670918	0.365843
2	0.642889	0.852798	0.843406	0.670929	0.365915
3	0.642888	0.852799	0.843406	0.670929	0.365917
4	0.642888	0.852799	0.843406	0.670929	0.365917
5	0.642888	0.852799	0.843406	0.670929	0.365917
$Kh_2 = 0.2, \quad a/h_2 = 0.3$					
	$h_1/h_2 = 0.1$	$h_1/h_2 = 0.3$	$h_1/h_2 = 0.5$	$h_1/h_2 = 0.7$	$h_1/h_2 = 0.9$
N	$\tilde{A}$	$\tilde{A}$	$\tilde{A}$	$\tilde{A}$	$\tilde{A}$
0	0.641882	0.852981	0.844097	0.668763	0.362793
1	0.642899	0.854827	0.847021	0.672847	0.366022
2	0.642907	0.854852	0.847053	0.672886	0.366084
3	0.642908	0.854853	0.847056	0.672890	0.366088
4	0.642908	0.854853	0.847056	0.672890	0.366089
5	0.642908	0.854853	0.847056	0.672890	0.366089

The numerical solutions of the dispersion relation (3.15) produce the edge wave frequency t_0h_2 and are plotted against the shelf width a/h_2 in Figs 2 and 3 for some particular values of the other parameters. It is observed that as $a/h_2 \to 0$, the edge wave frequencies are quite large and as a/h_2 increases the edge wave frequencies decrease and ultimately tend to zero. These types of observations are quite expected.

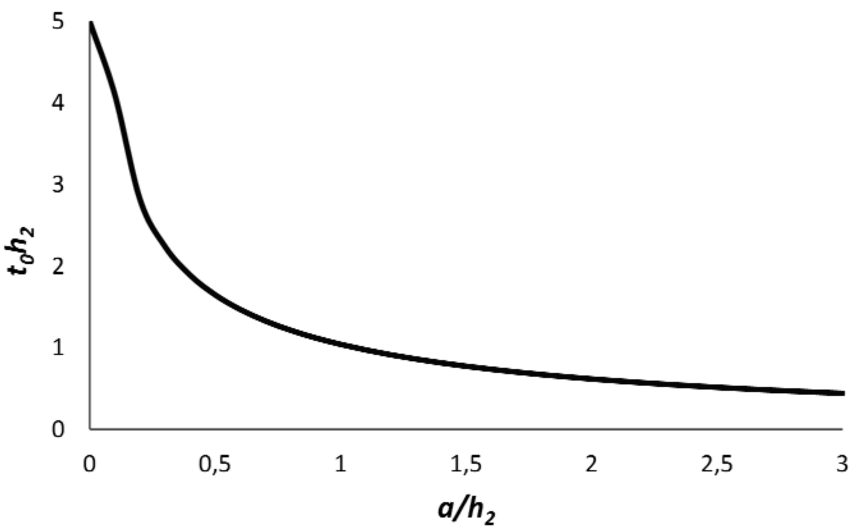

Fig.2. Frequency of edge waves for $Kh_2 = 0.2, \quad h_1/h_2 = 0.1$.

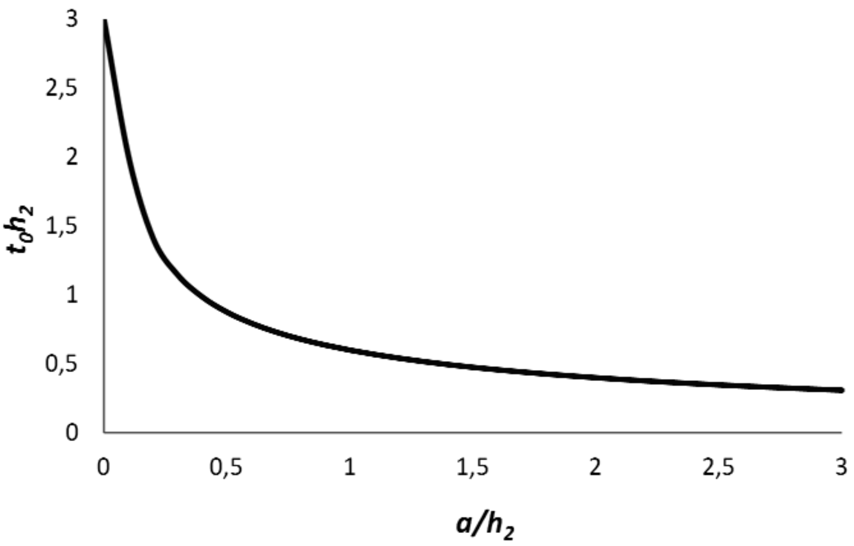

Fig.3. Frequency of edge waves for $Kh_2 = 0.2, \quad h_1/h_2 = 0.5$.

5. Conclusion

The existence of edge waves travelling along a submerged horizontal shelf is investigated in this paper. The method of multi-term Galerkin approximations in terms of ultra spherical Gegenbauer polynomials has been utilized to obtain very accurate numerical estimates for the integral involved in the dispersion relation of the problem considered here. By choosing only five terms in the Galerkin approximations, we achieve almost six figure accuracy in the numerical estimates of the integral. The numerical results are illustrated in a table and curves are presented showing the variation of frequency of the edge waves with the width of the shelf. Some expected known results are achieved.

Nomenclature

g – gravity
h_1, h_2 – depth of the shallow water
K, ϑ – wave number
t – time
x – horizontal distance
y – vertical distance
σ – wave frequency
φ – velocity potential

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