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A STUDY ON FRACTIONAL ORDER THERMOELASTIC HALF SPACE

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In this paper, we consider a one dimensional problem on a fractional order generalized thermoelasticity in half space subjected to an instantaneous heat source. The Laplace transform as well as eigen value approach techniques are applied to solve the governing equations of motion and heat conduction. Closed form solutions for displacement, temperature and stress are obtained and presented graphically.

Key word: thermoelasticity, instantaneous heat source, fractional order, thermoelastic half space, eigen value approach

1. Introduction

In 1956 Biot developed the dynamical coupled theory of thermoelasticity (CTE) which predicts an infinite speed of heat transportation in elastic media. The modified generalized thermoelastic theories have been developed by Lord and Shulman [1967] (LS model) and later Green and Naghdi [1991, 1992, 1993] (GN models I,II,III). Lord and Shulman tried to remove the paradox of infinite velocity of thermal disturbances inherent in the CTE. They used a wave type heat conduction law instead of the classical Fourier law and included a single relaxation time. Dhaliwal and Sherief [1980] extended the theory introducing the anisotropic case. In 1972, Green and Lindsay introduced two relaxation time parameters and modified the energy equation and constitutive equations in the theory of generalized thermoelasticity.

Abel first introduced fractional derivatives in the solution of integral equation which arises in the tautochrone problem. Fractional calculus was successfully used to modify many mathematical models in the field of mechanics of solids.

Kimmich [2002] considered anomalous diffusion characterized by the time-fractional diffusion wave equation using the Riemann-Lioville fractional integral.

We consider a one-dimensional problem for a half space in the context of the Lord and Shulman model with a heat source in fractional ordered generalized thermoelasticity. We have applied the eigen value approach and Laplace transform with numerical inversion. The obtained results are also presented graphically.

2. Basic equations and formulation of the problem

We consider a homogeneous, isotropic, thermoelastic conducting solid which is unstressed and unstrained initially, subjected to an instantaneous heat source. We consider the problem in a half-space

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region $R = \{x : 0 \le x < \infty\}$. The problem is to determine the subsequent distribution of temperature and deformation fields with regard to conductivity of the medium.

Catteneo [1958] introduced a law of heat conduction by modifying the classical Fourier law in the form

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -k\nabla T . \tag{2.1}$$

However, Youssef [2010] introduced another formula of heat conduction as

$$q_i + \tau_0 \frac{\partial^\alpha q_i}{\partial t^\alpha} = -kI^{\alpha-1}\nabla T , \qquad 0<\alpha\le 1 \tag{2.2}$$

where

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau , \qquad 0<\alpha\le 1, \tag{2.3}$$

$$I^0 f(t) = f(t) .$$

Here α is the fractional order parameter Youssef proved the uniqueness theorem and using the state-space approach presented one and two dimensional applications without any heat source term in the energy equation.

Now the basic heat equation is

$$kI^{\alpha-1}T_{,ii} = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T + \gamma T_0 e\right) - \rho Q - \rho\tau_0 \dot{Q} . \tag{2.4}$$

The constitutive equation takes the form

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - \gamma T\delta_{ij} . \tag{2.5}$$

The equation of motion without body forces takes the form

$$\sigma_{ij,j} = \rho\ddot{u}_i . \tag{2.6}$$

For a one dimensional medium, we assume that

$$u_x = u(x,t), \qquad u_y = u_z = 0. \tag{2.7}$$

The strain component is in the form

$$e = \frac{\partial u}{\partial x} . \tag{2.8}$$

Hence the heat equation, constitutive equation and equation of motion (2.4)-(2.6) may be written as

$$kI^{\alpha-1}\frac{\partial^2 T}{\partial t^2}=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T+\gamma T_0 e\right)-\rho\left(1+\tau_0\frac{\partial}{\partial t}\right)Q\,, \tag{2.9}$$

$$\left(\lambda+2\mu\right)\frac{\partial^2 u}{\partial x^2}-\gamma\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial x^2}\,, \tag{2.10}$$

$$\sigma_{xx}=\left(\lambda+2\mu\right)\frac{\partial u}{\partial x}-\gamma T\,. \tag{2.11}$$

The following non-dimensional variables are used as follows

$$\theta=\frac{T-T_0}{T_0}\,,\qquad x'=V\delta x,\qquad u'=V\delta u\,,\qquad t'=V^2\delta t\,,$$

$$\tau_0^{'}=V^2\delta\tau_0\,,\qquad \sigma_{xx}^{'}=\frac{\sigma_x}{\mu}\,,\qquad Q'=\frac{\rho Q}{kT_0V^2\delta^2}$$

where $$V^2=\frac{\lambda+2\mu}{\rho}\,,\qquad \beta^2=\frac{\lambda+2\mu}{\mu}\,,\qquad b=\frac{\gamma T_0}{\mu}\,,\qquad g=\frac{\gamma}{\rho C_E}\,,\qquad \delta=\frac{\rho C_E}{k}\,.$$

Therefore the Eqs (2.9)-(2.11) become

$$I^{\alpha-1}\frac{\partial^2\theta}{\partial t^2}=\left(\frac{\partial\theta}{\partial t}+\tau_0\frac{\partial^2\theta}{\partial t^2}\right)+g\left(\frac{\partial^2\theta}{\partial x\partial t}+\tau_0\frac{\partial^3\theta}{\partial x\partial t^2}\right)-\rho\left(1+\tau_0\frac{\partial}{\partial t}\right)Q\,, \tag{2.12}$$

$$\beta^2\frac{\partial^2 u}{\partial x^2}-b\frac{\partial\theta}{\partial x}=\beta^2\frac{\partial^2 u}{\partial t^2}\,, \tag{2.13}$$

$$\sigma=\beta^2\frac{\partial u}{\partial x}-b\left(1+\theta\right). \tag{2.14}$$

The half space $x\geq 0$ is subjected to an instantaneous heat source representing its energy continuously along positive direction of the x - axis with a constant velocity. So the heat source is taken as $Q=Q_0\delta\left(x-vt\right)$.
Where Q_0 is constant of heat sources and δ(.) is the Dirac delta function.

3. Solution procedure: the vector-matrix differential equation

Using the Laplace transform defined for any function $f\left(t\right)$ as follows

$$\overline{f}(x,s)=\int_0^\infty f(x,t)e^{-st}dt$$

where s is the transform parameter such that Re(s)>0 and applying both sides of Eqs (2.12)-(2.14) and assuming that $u,\frac{\partial u}{\partial t},\theta,\frac{\partial\theta}{\partial t}$ are equal to zero.

Equations (2.12)-(2.14) become

$$\overline{\sigma}=\beta^2\frac{\partial^2\overline{u}}{\partial x^2}-b\left(1+\overline{\theta}\right), \tag{3.1}$$

$$\frac{d^2\overline{u}}{dx^2}=s^2\overline{u}+a\frac{d\overline{\theta}}{dx}, \tag{3.2}$$

$$\frac{d^2\overline{\theta}}{dx^2}=s^\alpha\left(1+\tau_0 s\right)\overline{\theta}+gs^\alpha\left(1+\tau_0 s\right)\frac{d\overline{u}}{dx}-\frac{Q_0 s^{\alpha-1}\left(1+\tau_0 s\right)e^{-\frac{sx}{v}}}{v} \tag{3.3}$$

where $a=\frac{b}{\beta^2}$.

Equations (3.2) and (3.3) can be written in the form of a vector-matrix differential equation using (Lahiri *et al*. [2009])

$$\frac{d\overline{Z}}{dx}=A\overline{z}+\overline{f} \tag{3.4}$$

where

$$\overline{z}=\left(\overline{\theta},\overline{u},\frac{d\overline{\theta}}{dx},\frac{d\overline{u}}{dx}\right)^T, \tag{3.5}$$

$$A=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ a_{31} & 0 & 0 & a_{34}\\ 0 & a_{42} & a_{43} & 0\end{bmatrix}, \tag{3.6}$$

$$\overline{f}=\left(f_1,f_2,f_3,f_4\right)^T, \tag{3.7}$$

$$f_1=0,\quad f_2=0,\quad f_3=-\frac{Q_0 s^{\alpha-1}\left(1+\tau_0 s\right)e^{-\frac{sx}{v}}}{v},\quad f_4=0, \tag{3.8}$$

$$a_{31}=s^\alpha\left(1+\tau_0 s\right),\qquad a_{34}=gs^\alpha\left(1+\tau_0 s\right), \tag{3.9}$$

$$a_{42}=s^2\ , \qquad a_{43}=a\ . \tag{3.10}$$

The eigenvalues of the matrix A can be determined from the characteristic equation of the matrix A as

$$\lambda^4-\left[s^{\alpha}\left(1+\tau_0 s\right)\left(1+ga\right)+s^2\right]\lambda^2+s^{\alpha+2}\left(1+\tau_0 s\right)=0\ . \tag{3.11}$$

Therefore the coefficient matrix A has four eigenvalues which are λ_1, $\lambda_2,-\lambda_1,-\lambda_2$, respectively. The solution of the Eq.(3.11) is

$$\lambda_i^2=\frac{\left[s^{\alpha}\left(1+\tau_0 s\right)\left(1+ga\right)+s^2\right]\pm\sqrt{[s^{\alpha}\left(1+\tau_0 s\right)\left(1+ga\right)+s^2]^2-4s^{\alpha+2}\left(1+\tau_0 s\right)}}{2},\quad i=\pm 1,\quad \pm 2\ . \tag{3.12}$$

Now consider non-negative eigenvalues λ_1,λ_2 of A for physical nature of the problem.
Now the eigenvectors of the co-efficient matrix A corresponding to the eigenvalues λ_1, $\lambda_2,-\lambda_1,-\lambda_2$ of A be are

$$X_1=(X)_{\lambda=\lambda_1},\quad X_2=(X)_{\lambda=-\lambda_1},\quad X_3=(X)_{\lambda=\lambda_2},\quad X_4=(X)_{\lambda=-\lambda_2} \tag{3.13}$$

where

$$X=\left[\lambda^2-s^2,a\lambda,\lambda\left(\lambda^2-s^2\right),a\lambda^2\right]. \tag{3.14}$$

We consider the inverse of the matrix $V=\left(X_1,X_2,X_3,X_4\right)$ as $V^{-1}=\left(\omega_{ij}\right),\ i,\ j=1,2,3,4$
Then the solution of the differential Eq.(3.4) is

$$\bar{z}=\sum_{i=1}^{4}X_i y_i \tag{3.15}$$

where $$y_r=C_r e^{\lambda_r x}+e^{\lambda_r x}\int_{\infty}^{-\infty}\Omega_r e^{-\lambda_r x} \tag{3.16}$$

and $$\Omega_r=\sum_{j=1}^{4}\omega_{rj}f_j \tag{3.17}$$

where C_r is an arbitrary constant which is to be evaluated using boundary conditions.
Now using Eqs (3.15) and (3.17) the displacement component, heat component can be written as

$$\bar{u}\left(x,s\right)=C_1\lambda_1 e^{\lambda_1 x}+C_2\lambda_2 e^{\lambda_2 x}-aQ_1 e^{-\frac{sx}{\upsilon}}\left[\frac{\lambda_1}{\lambda_1+\frac{s}{\nu}}+\frac{\lambda_2}{\lambda_2+\frac{s}{\nu}}\right], \tag{3.18}$$

$$\bar{\theta}(x,s) = C_1\left(\lambda_1^2 - s^2\right)e^{\lambda_1 x} + C_2\left(\lambda_2^2 - s^2\right)e^{\lambda_2 x} - Q_1 e^{-\frac{sx}{\upsilon}}\left[\frac{\left(\lambda_1^2 - s^2\right)}{\lambda_1 + \frac{s}{\nu}} + \frac{\left(\lambda_2^2 - s^2\right)}{\lambda_2 + \frac{s}{\nu}}\right], \tag{3.19}$$

$$\bar{\sigma}(x,s) = b\left[C_1 s^2 e^{\lambda_1 x} + C_2 s^2 e^{\lambda_2 x} - 1 - Q_1 e^{-\frac{sx}{\upsilon}}\left[\left[\frac{\left(\lambda_1^2 - s^2\right)}{\lambda_1 + \frac{s}{\nu}} + \frac{\left(\lambda_2^2 - s^2\right)}{\lambda_2 + \frac{s}{\nu}}\right] - \frac{s}{\nu}\left[\frac{\lambda_1}{\lambda_1 + \frac{s}{\nu}} + \frac{\lambda_2}{\lambda_2 + \frac{s}{\nu}}\right]\right]\right] \tag{3.20}$$

where

$Q_1 = -\dfrac{Q_0 s^{\alpha-1}\left(1 + \tau_0 s\right)}{2v\lambda_1\left(\lambda_1^2 - \lambda_2^2\right)}$ and C_1, C_2 are arbitrary constants.

4. Initial and boundary conditions

We consider two boundary conditions for the half-space problem to obtain the co-efficients C_1 and C_2 of Eqs (3.18)-(3.20).

CASE-1:

(a) mechanical boundary condition- $\sigma(0,t) = 0$,

(b) thermal boundary condition- $\theta(0,t) = \theta_0 H(t)$

where θ_0 is the constant temperature and $H(t)$ is the Heaviside unit step function. The Laplace transformed boundary conditions, for $t > 0$ are

$$\bar{\sigma}(0,s) = 0, \tag{4.1}$$

$$\bar{\theta}(0,s) = \frac{\theta_0}{s} \tag{4.2}$$

Now from Eqs (3.18)-(3.20) we get

$$C_1 = \frac{\left(\lambda_2^2 - s^2\right)R - M}{\left(\lambda_2^2 - \lambda_1^2\right)} \quad \text{and} \quad C_2 = \frac{M - \left(\lambda_1^2 - s^2\right)R}{\left(\lambda_2^2 - \lambda_1^2\right)} \tag{4.3}$$

where $$R = \frac{1}{s^2} - \frac{Q_0 s^{\alpha-3}\left(1 + \tau_0 s\right)}{2v\lambda_1\left(\lambda_1^2 - \lambda_2^2\right)}\left[\left\{\frac{\left(\lambda_1^2 - s^2\right)}{\lambda_1 + \frac{s}{\nu}} + \frac{\left(\lambda_2^2 - s^2\right)}{\lambda_2 + \frac{s}{\nu}}\right\} + \frac{s}{v}\left\{\frac{\lambda_1}{\lambda_1 + \frac{s}{\nu}} + \frac{\lambda_2}{\lambda_2 + \frac{s}{\nu}}\right\}\right]$$

and $$M=\frac{\theta_0}{s}-\frac{Q_0 s^{\alpha-1}(1+\tau_0 s)}{2v\lambda_1\left(\lambda_1^2-\lambda_2^2\right)}\left[\frac{\left(\lambda_1^2-s^2\right)}{\lambda_1+\frac{s}{\nu}}+\frac{\left(\lambda_2^2-s^2\right)}{\lambda_2+\frac{s}{\nu}}\right].$$

CASE-2:

(a) mechanical boundary condition - $\sigma(0,t)=0$,

(b) thermal boundary condition - $$\theta(0,t)=\begin{cases}0, t\le 0\\ \theta_1\frac{t}{t_0}, 0<t\le t_0\\ \theta_1, t>t_0\end{cases}$$

where θ_1 is a constant and t_0 is the ramping time parameter.

Taking the Laplace transformation of the boundary condition, we get

$$\overline{\sigma}(0,s)=0, \tag{4.4}$$

$$\overline{\theta}(0,s)=\frac{\theta_1}{t_0}\left(\frac{1-e^{-t_0 s}}{s^2}\right). \tag{4.5}$$

Similarly, from Eqs (3.18)-(3.20)we get

$$C_1=\frac{\left(\lambda_2^2-s^2\right)R-M}{\left(\lambda_2^2-\lambda_1^2\right)} \quad \text{and} \quad C_2=\frac{M-\left(\lambda_1^2-s^2\right)R}{\left(\lambda_2^2-\lambda_1^2\right)} \tag{4.5}$$

where $$R=\frac{1}{s^2}-\frac{Q_0 s^{\alpha-3}(1+\tau_0 s)}{2v\lambda_1\left(\lambda_1^2-\lambda_2^2\right)}\left[\left\{\frac{\left(\lambda_1^2-s^2\right)}{\lambda_1+\frac{s}{\nu}}+\frac{\left(\lambda_2^2-s^2\right)}{\lambda_2+\frac{s}{\nu}}\right\}+\frac{s}{v}\left\{\frac{\lambda_1}{\lambda_1+\frac{s}{\nu}}+\frac{\lambda_2}{\lambda_2+\frac{s}{\nu}}\right\}\right]$$

and $$M=\frac{\theta_1}{t_0}\left(\frac{1-e^{-t_0 s}}{s^2}\right)-\frac{Q_0 s^{\alpha-1}(1+\tau_0 s)}{2v\lambda_1\left(\lambda_1^2-\lambda_2^2\right)}\left[\frac{\left(\lambda_1^2-s^2\right)}{\lambda_1+\frac{s}{\nu}}+\frac{\left(\lambda_2^2-s^2\right)}{\lambda_2+\frac{s}{\nu}}\right].$$

5. Numerical representation

The inversion of the Laplace transform of the expressions given in Eqs (3.18)-(3.20) for displacement, temperature and stress respectively in the space-time domain is very complex. So we developed an efficient computer programme for the inversion of the Laplace transforms. We follow the method of Bellman *et al*. (1966) for this inversion of the Laplace transform. The numerical computations for the field variables are performed for the time instants

$$t_1=0.025775,\quad t_2=0.138382,\quad t_3=0.352509,\quad t_4=0.693147,$$

$$t_5=1.21376,\quad t_6=2.04612,\quad t_7=3.67119,$$

which are the roots of the Legendre polynomial of degree seven (vide Bellman *et al*. (1966)).

Here copper is taken as the thermoelastic material and the parameters (in S.I units) are given below (Youssef, 2006)

$$\lambda=7.76\times 10^{10}\,N/m^2, \mu=3.86\times 10^{10}\,N/m^2, \quad \tau_0=0.001,$$

$$T_0=293\,K, \quad k=386\,N/Ks, \quad \alpha_T=1.78\times 10^{-5}\,K^{-1},$$

$$v=0.08, \quad \delta=888.6\,m/s^2, \rho=8954\,Kg/m^3,$$

$$\beta^2=4, \quad \epsilon=0.0168.$$

6. Graphical representation

Now we consider the graphs obtained for two cases.

CASE-1:

(i) Figure 1: This figure shows the variation of displacement against the space variable x for $\alpha=0.1$. The displacement u attains its maximum value near $t=t_6$ for $x=0$ and then decreases rapidly at $t=t_4$ and $t=t_2$.

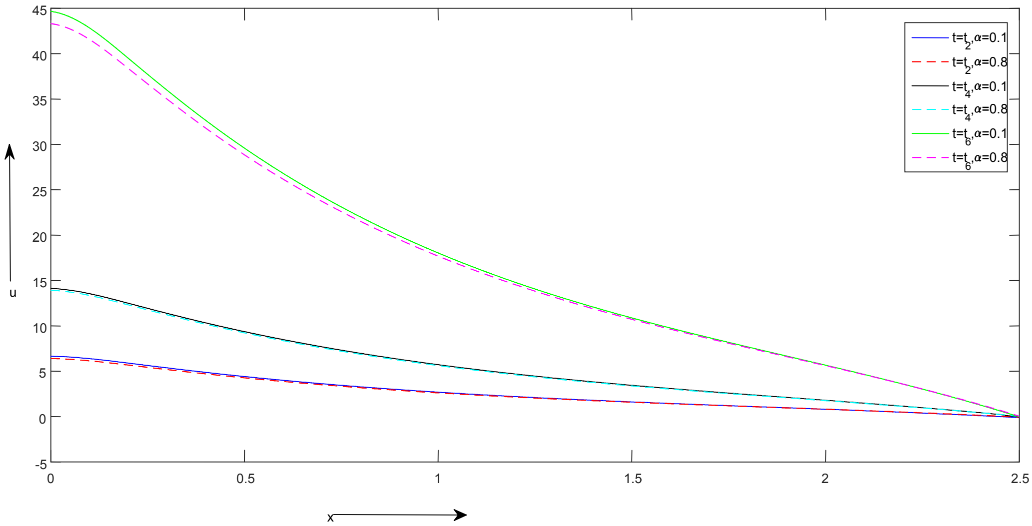

Fig.1. Displacement distribution against x.

(ii) Figure 2:This figure shows that the temperature distribution against the space variable x for $\alpha=0.1$ attains its maximum value near $t=t_7$ for $x=0.2$ and then decreases rapidly. Similarly near $t=t_5$ the temperature distribution increases upto $x=0.2$ and thereafter decreases towards zero. But for $t=t_2$ it decreases upto $x=0.4$ thereafter it increases and converges to zero. At $\alpha=0.8$ the graphs are similar but the attained maximum values and minimum values are different.

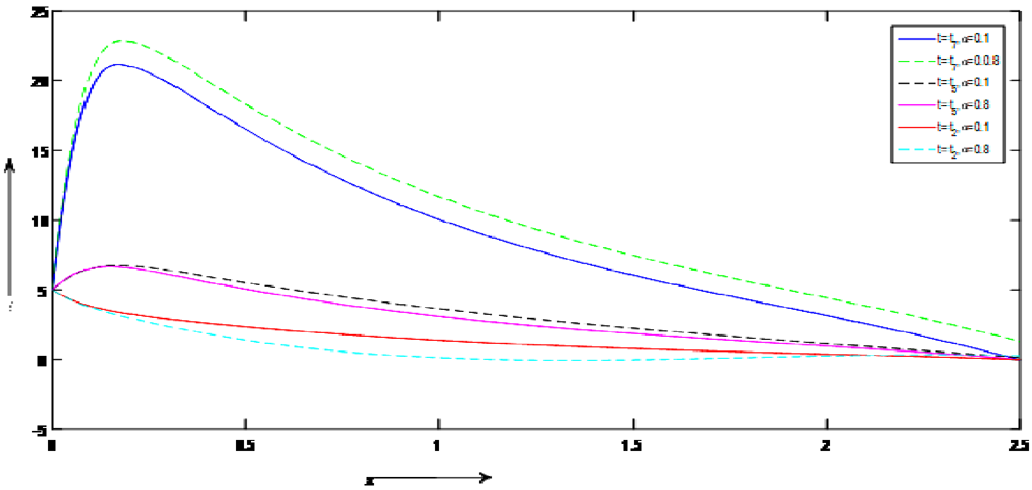

Fig.2. Temperature distribution against *x*.

(iii) Figure 3: This figure shows that the stress distribution against the space variable x for $\alpha = 0.1$ attains its maximum value near $t = t_7$ for $x = 0.2$ and then decreases rapidly. Similarly, for $t = t_5$ and $t = t_1$ the displacement distribution at first increases and thereafter decreases towards zero. For $\alpha = 0.8$ the graphs are similar but their attained peak points and lowest points are.

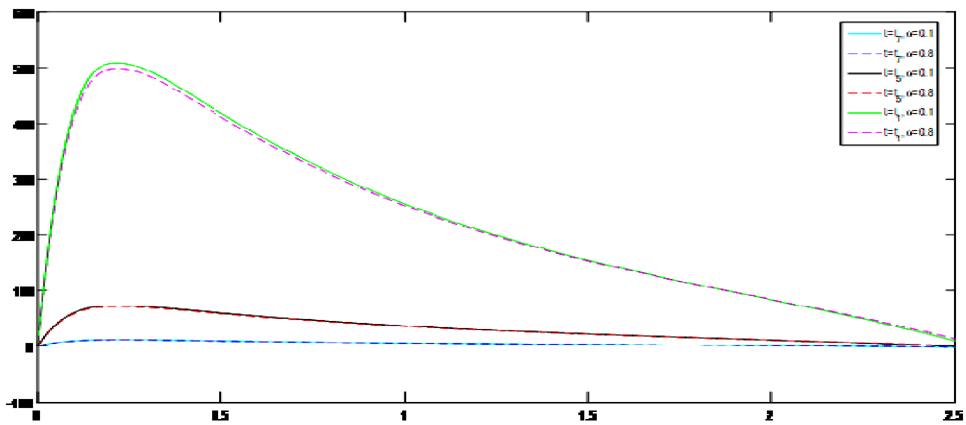

Fig.3. Stress distribution against *x*.

CASE-2:

(iv) Figure 4: The variation of displacement against the space variable x shows a difference in values for $\alpha = 0.1$ and $\alpha = 0.8$ in the range $1 \le t \le 7$. For $\alpha = 0.1$ the displacement *u* attains its maximum value near $= t_2$, $t = t_4$, $t = t_7$ for $x = 0$ and then decreases towards zero. For $\alpha = 0.8$ the temperature distribution also starts from the maximum values near $= t_2$, $t = t_4$, $t = t_7$.

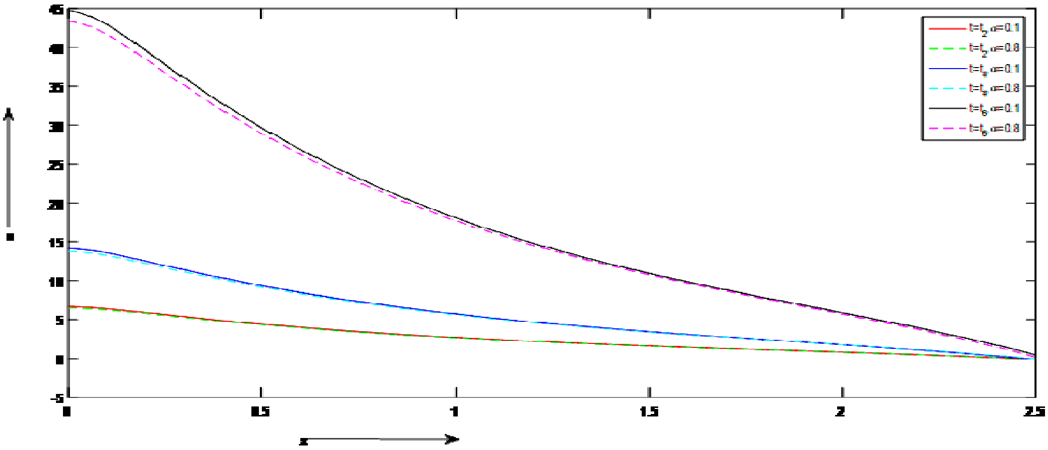

Fig.4. Displacement distribution *u* against *x*.

(v) Figure 5: For $\alpha = 0.1$ the temperature θ increases and attains its maximum value near $t = t_7, t_5, t_3$ for $x = 0.3$ and then decreases rapidly but for $\alpha = 0.8$ the graph shows different peak points.

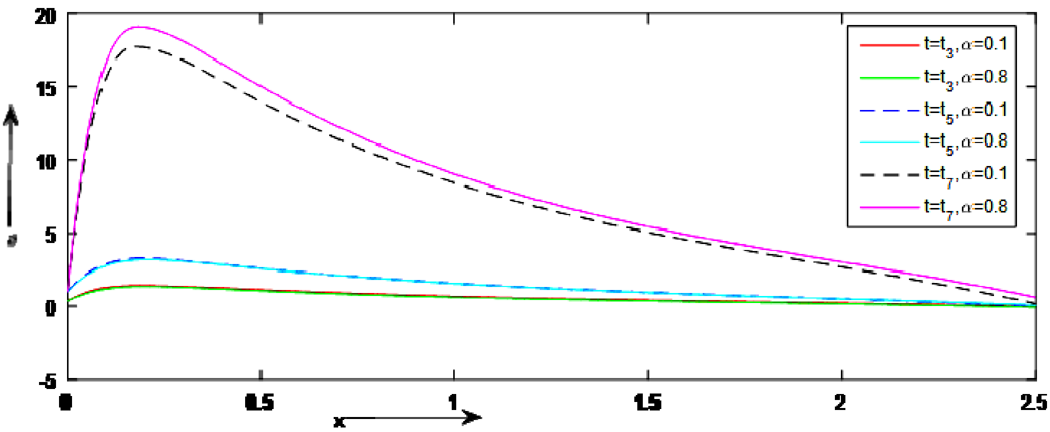

Fig.6. Temperature distribution against *x*.

(vi) Figure 6: This figure shows the stress distribution against the space variable *x*. For $\alpha = 0.1$ and $\alpha = 0.8$ the value of stress attains its maximum near $t = t_7$ $, t_5$, t_1 for $x = 0.2$ and thereafter decreases rapidly.

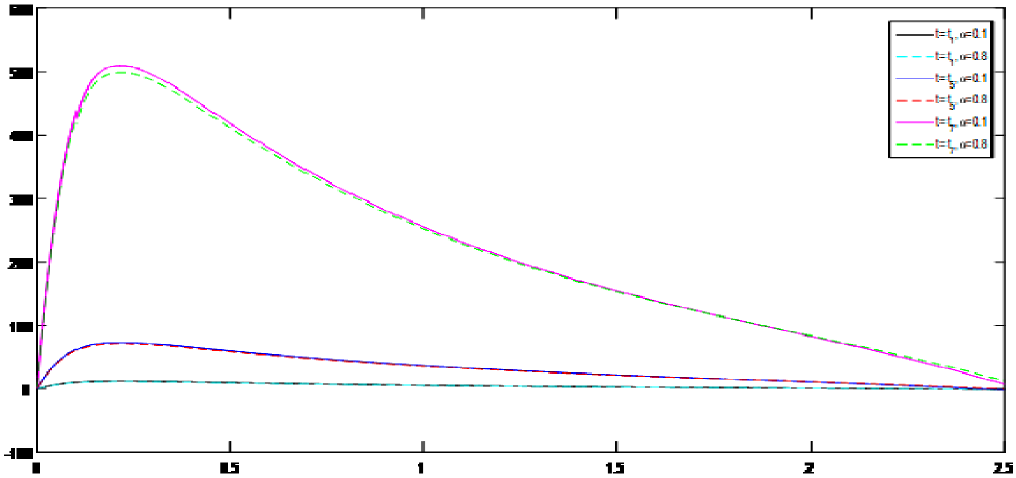

Fig.6. The stress distribution against *x*.

7. Conclusion

The following conclusions can be drawn from the study on the thermoelastic solid subjected to an instantaneous heat source in the context of fractional order theory of thermoelasticity:

1. The non-dimensional temperature reaches its maximum value at the location $x = vt$ for specified time.
2. The magnitude of the maximum value of temperature increases as the fractional order parameter increases.

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Conflict of interest

The authors don't have any conflict of interest.

Nomenclature

C_E – specific heat
e_{ij} – components of strain tensor
k – thermal conductivity
T – temperature
T_0 – reference temperature
u_i – displacement components
q_i – heat flux components
V – longitudinal wave speed
v – the velocity of the heat source
α_T – thermal expansion coefficient
δ – thermal viscosity
λ,μ – Lame's constants

γ $-$ $\alpha_T(3+2)$

ε $-$ $\dfrac{\gamma}{C_E}$ dimensionless coupling constant

ρ $-$ density

τ_0 $-$ relaxation times

σ_{ij} $-$ components of stress tensor

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