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COMPUTER ANALYSIS OF DYNAMIC RELIABILITY OF SOME CONCRETE BEAM STRUCTURE EXHIBITING RANDOM DAMPING

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An efficiency of the generalized tenth order stochastic perturbation technique in determination of the basic probabilistic characteristics of up to the fourth order of dynamic response of Euler-Bernoulli beams with Gaussian uncertain damping is verified in this work. This is done on civil engineering application of a two-bay reinforced concrete beam using the Stochastic Finite Element Method implementation and its contrast with traditional Monte-Carlo simulation based Finite Element Method study and also with the semi-analytical probabilistic approach. The special purpose numerical implementation of the entire Stochastic perturbation-based Finite Element Method has been entirely programmed in computer algebra system MAPLE 2019 using Runge-Kutta-Fehlberg method. Further usage of the proposed technique to analyze stochastic reliability of the given structure subjected to dynamic oscillatory excitation is also included and discussed here because of a complete lack of the additional detailed demands in the current European designing codes.

Key words: reliability analysis, Stochastic Finite Element Method, generalized stochastic perturbation technique, forced vibrations; Runge-Kutta-Fehlberg method.

1. Introduction

Structural vibration is a matter of great concern in reference to reliability, durability as well as serviceability of structures and multiple guidelines can be found in order to satisfy requirements of Ultimate or Serviceability Limit States. Mayers, for instance, described in [1] that DIN 4150 [2] recalls requirements of peak vibration velocity of structure, which guarantees maintenance of structural integrity for industrial buildings. Other criteria can be found in AS2625, which provides figures for vibration analysis referring to a reduction of excessive maintenance of machinery. An acceptable vibration level, in reference to frequency, displacements velocity and acceleration in a given structure strictly depends of course upon its final purpose. For that reason the Australian Standard AS2670 [3] brings some guidelines to estimate an impact of vibrations on human comfort depending on the purpose of a structure (different for residential area, workshops, offices or critical working areas like operating-theatres and high precision laboratories). The German Standard DIN 4150 brings limitations for vibration velocity for given frequency of dynamic load and type of structure. and maximum peak velocity equals 40 mm/sec for inter-floor ceiling structure classified as an industrial area.

It is well known that the reliability index [4] in civil engineering structural analysis may approach its minimum admissible limits provided in the engineering design codes when expected values of displacements are too large and lead to structural failure, when expected values of deformation rates are too large and also when their standard deviations are too high, so that structural response becomes almost unpredictable [5-7]. It is clear that this last effect cannot be estimated in classical structural dynamics. Therefore, precise determination of the first two probabilistic characteristics of structural dynamic response is so important and

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needs to be efficiently estimated and this has been done using a number of various theoretical and numerical techniques [8-14]. Further, one would approximate probability density function of structural response at the given discrete time of vibrations time interval, therefore the calculation of third and fourth order probabilistic characteristics (like skewness and kurtosis) remains also very important [14,15]. Let us note that there is no corresponding stochastic software based on the additional expansion of the traditional Finite Element Method for vibration analysis [16,17]. This work presents some numerical approach to analyse Euler-Bernoulli beams with random damping under user-defined dynamic load spectrum.

Table 1. Requirements to structural vibration of DIN4150.

Type of structure	Peak Particle Velocity PPV (mm/sec)			
	At foundation level Frequency range (Hz)			At floor level of top most story (all frequencies)
	< 10	*10 – 50*	*50 - 100*	
(i) Building used as offices & industrial structures	*20*	*20-40*	*40-50*	*40*
(ii) Domestic houses & associated constructions, structures with plasters	*5*	*5-15*	*15-20*	*15*
Buildings which do not fall under (i) & (ii) and objects of historic interest or other sensitive structures	*3*	*3-8*	*8-10*	*8*

There are several well-established theoretical and computer methods enabling the determination of stochastic dynamic response of civil engineering structures. As it is expected, analytical models are rather scarce and available for simple static schemes. Undoubtedly, it can be precisely done using a family of the Monte-Carlo simulation techniques, but it demands relatively large computer power and usually causes huge time consumption. Secondly, Karhunen-Loeve and polynomial chaos expansion methods have been developed, but their minor aspect is a relatively easy determination of the first two probabilistic moments only. Very recently Probability Transformation Method has been worked out to directly recover resulting PDF of structural dynamic response on the basis of the input one. However, this approach is difficult in common implementation with traditional engineering FEM software. Frequently, stochastic perturbation method employed in this work is simply preferred. The generalized tenth order stochastic perturbation technique has been chosen, which allows an exact determination of the first two probabilistic moments and precise enough computation of third and fourth order probabilistic characteristics. Iterative version of this method is used here, where derivation of the perturbation-based formulas does not treat expectations as equal to the mean values of structural responses as in its linearized version. Contrary to many lower order perturbation-based approaches and computations implemented using the Direct Differentiation Method (DDM), where hierarchical equations of increasing order are formed and solved, higher order approach proposed here uses the Least Squares Method (LSM) approximations of structure displacements with polynomial basis. These approximations are based upon several FEM tests solved for gradually increased value of structural damping. These LSM fittings have been also programmed in computer algebra system MAPLE 2019 to have the same computational environment for further numerical experiments. Computer code implemented includes Finite Element Method procedures together with the fourth order Runge-Kutta-Fehlberg method to solve matrix equations of motion. It should be mentioned that very good coincidences of displacement determined using the proposed computer method with the Monte-Carlo simulation-based statistical estimators and with probabilistic characteristics resulting from the semi-analytical method have been obtained. That is why these characteristics returned by the iterative generalized stochastic perturbation method may be efficiently used in further reliability assessment. Computer code presented in this work will be extended next towards interoperability with some existing FEM software to analyse structural reliability of civil engineering structures with uncertainty subjected to practically important vibrations.

2. Governing equations and computational implementation

An elastic multi-bay Euler-Bernoulli beam made of the steel reinforced concrete is analysed in this work. It is excited by sinusoidal dynamic forces applied in the halves of the bays and its uncertain damping results in a stochastic time-dependent response, whose basic moments and characteristics are to be computed. These characteristics for displacements and velocities (necessary in Serviceability Limit States) need to be evaluated considering final assessment of the reliability indices. Since any analytical technique cannot serve for this purpose, three different computational techniques have been employed to achieve this goal and to determine expectations, variances, skewness and kurtosis for structural vibrations. This analysis is completed here using the Runge-Kutta-Fehlberg method, while stochastic output is obtained using polynomial dynamic response functions recovered via the Least Squares Method approximations of structural behaviour.

Mathematical models start from classical equations of motion relevant to the elasto-dynamic system and they are discretized using variational formulation of the Finite Element Method. A two-dimensional and two-noded beam element corresponding to the Euler-Bernoulli theory has been employed in further analysis, so that discretization of transverse deformations and rotation angles undergoes via linear shape functions; some other approaches can be found in [18]. It is well known that the matrix equation describing dynamic equilibrium in the Finite Element Method language can be rewritten as follows [16,17]:

$$[M]\cdot\ddot{\mathrm{y}}+[C]\cdot\dot{\mathrm{y}}+[K]\cdot\mathrm{y}=\mathrm{F} \tag{2.1}$$

where stiffness and mass matrices for a single finite element before aggregation are described as follows:

$$\left[k^{(e)}\right]=\frac{EJ}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \tag{2.2}$$

$$\left[m^{(e)}\right]=\frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}. \tag{2.3}$$

Parameters E, A, J denote in turn the Young modulus, cross-sectional area and inertia modulus, while L stands for the finite element length. The Damping matrix is created using a decomposition into the stiffness and mass matrices contributions and Rayleigh coefficients $\hat{\alpha}$ and $\hat{\beta}$ as follows:

$$[C]=\hat{\alpha}\cdot[M]+\hat{\beta}\cdot[K]; \tag{2.4}$$

these damping factors can be found experimentally and they have been calculated using effective viscous damping coefficient according to the model described in [4]; its unit value is equivalent to critical damping.

Equation (2.1) has been solved with the use of the 4^{th} order Runge-Kutta-Fehlberg scheme [19,20]. Let us consider as an illustration the following first order ordinary differential equation:

$$y'(x) = f(x, y) . \tag{2.5}$$

Its solution is described by a set of the following discrete values:

$$y_{i+1} = y_i + \frac{1}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4) \tag{2.6}$$

where y_{i+1} and y_i are values of the given function in current and previous step of computation. The symbols k_1, k_2, k_3, k_4 are defined here as

$$k_1 = h \cdot f(x_i, y_i) , \tag{2.7}$$

$$k_2 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right), \tag{2.8}$$

$$k_3 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right), \tag{2.9}$$

$$k_4 = h \cdot f(x_i + h, y_i + k_3) . \tag{2.10}$$

Variable h denotes here the assumed increment of the method, while index i describes step number. The 4th order Runge-Kutta-Fehlberg Finite Element Method application in the analysed structural dynamics problems refers to the following governing equation of motion:

$$\ddot{\boldsymbol{y}} = [M]^{-1} \cdot \left[\boldsymbol{F} - [C] \cdot \dot{\boldsymbol{y}} - [K] \cdot \boldsymbol{y}\right]. \tag{2.11}$$

The right-hand side of this equation can be represented

$$\ddot{\boldsymbol{y}} = [M]^{-1} \cdot f(y, \dot{y}, t) . \tag{2.12}$$

The acceleration vector in the Runge-Kutta-Fehlberg method (RKF45) is described at the time t_s and time increment h time step as

$$\ddot{\boldsymbol{y}}_{\boldsymbol{ts}} = [M]^{-1} \cdot f(y_{ts}, \dot{y}_{ts}, t_s) . \tag{2.13}$$

Sequentially, we determined the displacement and velocity at time $t_s + {}^{3}\!/_{8} h$ as

$$\begin{aligned} \overline{\boldsymbol{y}}_{ts+\frac{3}{8}h} &= \boldsymbol{y}_{ts} + \frac{3}{32} h \cdot \dot{\boldsymbol{y}}_{ts} + \frac{9}{32} h \cdot \overline{\dot{\boldsymbol{y}}}_{ts+\frac{1}{4}h} , \\ \overline{\dot{\boldsymbol{y}}}_{ts+\frac{3}{8}h} &= \dot{\boldsymbol{y}}_{ts} + \frac{3}{32} h \cdot \ddot{\boldsymbol{y}}_{ts} + \frac{9}{32} h \cdot \overline{\ddot{\boldsymbol{y}}}_{ts+\frac{1}{4}h} \end{aligned} \tag{2.14}$$

and acceleration at the same time is found as

$$\ddot{\bar{y}}_{ts+\frac{3}{8}h} = [M]^{-1} \cdot f\left(\bar{y}_{ts+\frac{3}{8}h}, \dot{\bar{y}}_{ts+\frac{3}{8}h}, t_s + \frac{3}{8}h \right). \tag{2.15}$$

Further, displacement and velocity at time $t_s + {}^{12}\!/_{13}h$ are calculated as

$$\bar{y}_{ts+\frac{12}{13}h} = y_{ts} + \frac{1932}{2197}h \cdot \dot{y}_{ts} - \frac{7200}{2197}h \cdot \dot{\bar{y}}_{ts+\frac{1}{4}h} + \frac{7296}{2197}h \cdot \dot{\bar{y}}_{ts+\frac{3}{8}h}, \tag{2.16}$$

$$\dot{\bar{y}}_{ts+\frac{12}{13}h} = \dot{y}_{ts} + \frac{1932}{2197}h \cdot \ddot{y}_{ts} - \frac{7200}{2197}h \cdot \ddot{\bar{y}}_{ts+\frac{1}{4}h} + \frac{7296}{2197}h \cdot \ddot{\bar{y}}_{ts+\frac{3}{8}h}, \tag{2.17}$$

whereas acceleration at time $t_s + {}^{12}\!/_{13}h$ is determined as

$$\ddot{\bar{y}}_{ts+\frac{12}{13}h} = [M]^{-1} \cdot f\left(\bar{y}_{ts+\frac{12}{13}h}, \dot{\bar{y}}_{ts+\frac{12}{13}h}, t_s + \frac{12}{13}h \right). \tag{2.18}$$

Finally, one calculates displacement and velocity at time $t_s + h$ and there holds

$$\bar{y}_{ts+h} = y_{ts} + \frac{439}{216}h \cdot \dot{y}_{ts} - 8h \cdot \dot{\bar{y}}_{ts+\frac{1}{4}h} + \frac{3680}{513}h \cdot \dot{\bar{y}}_{ts+\frac{3}{8}h} - \frac{845}{4104} \cdot \dot{\bar{y}}_{ts+\frac{12}{13}}, \tag{2.19}$$

$$\dot{\bar{y}}_{ts+h} = \dot{y}_{ts} + \frac{439}{216}h \cdot \ddot{y}_{ts} - 8h \cdot \ddot{\bar{y}}_{ts+\frac{1}{4}h} + \frac{3680}{513}h \cdot \ddot{\bar{y}}_{ts+\frac{3}{8}h} - \frac{845}{4104} \cdot \ddot{\bar{y}}_{ts+\frac{12}{13}}. \tag{2.20}$$

Acceleration at the same time equals to

$$\ddot{\bar{y}}_{ts+h} = [M]^{-1} \cdot f\left(\bar{y}_{ts+h}, \dot{\bar{y}}_{ts+h}, t_s + h \right). \tag{2.21}$$

The final values of displacement, velocity and acceleration at $t_s + h$ are represented by the following equations:

$$y_{ts+h} = y_{ts} + h \cdot \left(\frac{25}{216} \cdot \dot{y}_{ts} + \frac{1408}{2565} \cdot \dot{\bar{y}}_{ts+\frac{3}{8}h} + \frac{2197}{4104} \cdot \dot{\bar{y}}_{ts+\frac{12}{13}} - \frac{1}{5} \cdot \dot{\bar{y}}_{ts+h} \right), \tag{2.22}$$

$$\dot{y}_{ts+h} = \dot{y}_{ts} + h \cdot \left(\frac{25}{216} \cdot \ddot{y}_{ts} + \frac{1408}{2565} \cdot \ddot{\bar{y}}_{ts+\frac{3}{8}h} + \frac{2197}{4104} \cdot \ddot{\bar{y}}_{ts+\frac{12}{13}} - \frac{1}{5} \cdot \ddot{\bar{y}}_{ts+h} \right), \tag{2.23}$$

$$\ddot{y}_{ts+h} = [M]^{-1} \cdot f\left(y_{ts+h}, \dot{y}_{ts+h}, t_s + h \right). \tag{2.24}$$

This approach has been entirely implemented in the symbolic algebra system MAPLE and has been distributed into three different sections. The first section is dedicated to formulation and solving a governing equation of forced and damped vibration of structure. Formulation of the problem is performed with the Finite Element Method and the governing equation is therefore solved with the RKF algorithm presented above. The second section creates Structure-Response Function (SRF) from the discrete values of displacements and velocity as they are a direct product of computational procedure from the previous step. The final part of the algorithm involves a probabilistic approach to the variable of SRF with computations on its statistic values and therefore performs reliability index estimation of non-stationary problem.

To pursue the previously appointed goal, our algorithm requires several input data, which afterward are utilized automatically. From the perspective of structural geometry, length, static diagram and moment of inertia (cross-section characteristic) are required. Material data provide elasticity modulus and mass of construction per unit of length. It is necessary to mention that concrete cross-section includes some small cracks and their impact on the structural behaviour has been provided by an application of the substitutional cross-section of the beam, where the original cross-sectional area of steel rebars has been multiplied by an additional factor, concerning the relation between the Young modulus of steel and concrete. Then, the cross-sectional mass center and inertia moment of the original non-cracked beam have been modified. A normal force driven by the reinforcing rebars in the tensioned area has been calculated to get full information about the size of the compressed cross-sectional area in the Ultimate Limited State. Further, it has been assumed that this compressed part of the cross-section affects final calculations, so that a moment of inertia after cracking has been calculated as a product of this compressed area and cross-section of rebars reinforcement in the tensioned area multiplied by an alpha factor. A relation between the inertia moment after and before cracking of the original homogenous concrete has been calculated as equal to *0,254*, which means that approximately *75%* of the initial beam bending stiffness has been lost during exploitation.

It was also a necessary to identify the dynamic load characteristics with respect to: the amplitude and frequency of dynamic force as well as mass and location of the dynamic force generator (device). Moreover, for FEM implementation: the number of finite elements along with length of each element, boundary conditions for stiffness, mass and damping matrices with respect to static diagram and additional (often individual) data for purposes of damping matrix definition are obligatory. In addition, a time step for the RKF method is required in order to perform calculations. From the probabilistic point of view it is also a necessary to assume characteristics of random variables: distribution, expected value and coefficient of variation.

Dividing the algorithm into three main sections was compulsory for maintaining a proper order and also ensuring fluent and faster computations. Calculations conducted in one section are exported to the file directory and then imported to the following section of algorithm. This operation was implemented also as a solution to a long-lasting calculations problem of the RFK algorithm for the modified governing equation.

The first four probabilistic characteristics of the structural dynamic response of the given beam have been determined. Expected values of coefficients of variation, skewness and kurtosis have been determined in the same time period as the deterministic counterparts of displacements, velocities as well as accelerations. These characteristics have been determined with the use of polynomial response functions of random damping b as

$$y_k\left(\hat{t}\right)=\sum_{i=1}^{n}C_{ki}\left(\hat{t}\right)b^i . \tag{2.25}$$

The coefficients C_{ki} have been determined all using the Least Squares Method fitting to the given initial series of FEM experiments with deterministically modified mean value of b itself. A precision of this fitting may remarkably depend upon the chosen time incrementation during the FEM vibration computer analysis. Three concurrent probabilistic techniques have been finally engaged, which have statistical, analytical and expansion nature and these are the Monte-Carlo simulation, semi-analytical as well as stochastic generalized perturbation technique. They are all based on the definition of the expected value of the response function $y_k(b)$

$$E\left(y_k\left(b,\hat{t}\right)\right)=\int_{-\infty}^{\infty} y_k\left(b,\hat{t}\right)p_b\left(x\right)dx \equiv \frac{1}{N}\sum_{j=1}^{N} y_k^j\left(b,\hat{t}\right), \tag{2.26}$$

where N denotes the total number of random samples in the Monte-Carlo approach, j indices current random trial obtained for the given discrete time $\hat{t}$. Fundamental differences in-between these three methods may be seen for variance of the structural displacements $y_k\left(b,\hat{t}\right)$. They can be defined in turn for statistical, analytical and expansion approaches as

$$Var_{MCS}\left(y_k\left(b,\hat{t}\right)\right)=\frac{1}{N-1}\sum_{j=1}^{N}\left(y_k^j\left(b,\hat{t}\right)-E\left[y_k\left(b,\hat{t}\right)\right]\right)^2, \tag{2.27}$$

$$Var_{SAM}\left(y_k\left(b,\hat{t}\right)\right)=\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n} C_{ki}\left(\hat{t}\right)b^i - E\left[y_k\left(b,\hat{t}\right)\right]\right)^2 pb\left(x\right)dx, \tag{2.28}$$

$$\begin{aligned} &Var_{SPT}\left(y_k\left(b,\hat{t}\right)\right)= \\ &=-\int_{-\infty}^{\infty}\left(y^0\left(b^0,\hat{t}\right)+\sum_{l=1}^{p}\frac{\varepsilon^l}{l!}\left.\frac{\partial^l y_k\left(b,\hat{t}\right)}{\partial b^l}\right|_{b=b^0}\cdot\left(b-b^0\right)^l - E\left[y_k\left(b,\hat{t}\right)\right]\right)^2 p_b\left(x\right)dx \end{aligned} \tag{2.29}$$

Parameters p and ε denote here the perturbation order as well as the perturbation parameter frequently adapted as equal to *1*. The last equation enables a clear explanation of a difference between the linearized and iterative method performed in this work. The linearized version is based upon the equity of zeroth order term and expected values, so the first and the last components simply vanish. A substitution of initially calculated expectation of $y_k(b,\hat{t})$ is used in the iterative approach, so that algebraic transformation is decisively more complex but brings higher accuracy for the third and fourth order probabilistic moments [14,23]. Finally, it should be mentioned that all these stochastic procedures together with automatic derivation of perturbation-based formulas have been programmed in the computer algebra system MAPLE 2019.

3. Numerical example

The model of a beam which represents a repeatable section of an orthotropic reinforced-concrete plate was taken into account as a research model. A segment of *2* meters width was treated as a starting point for calculating cross-sectional characteristic (Fig.1). Stiffness loss due to cracking of bending concrete element was taken under consideration while estimating the moment of inertia for a given cross-section; a difference in the Young modulus of concrete and steel reinforcing-bars was taken into account as well. It was assumed that the plate is considered to be of two-span, supported on rotation-free joints. Repeatable cross-section of the plate divided into *40* finite elements – each of *35.0 cm* length. A geometry, static scheme, nodal numbers, finite elements and location of dynamic forces are shown in Fig.3.

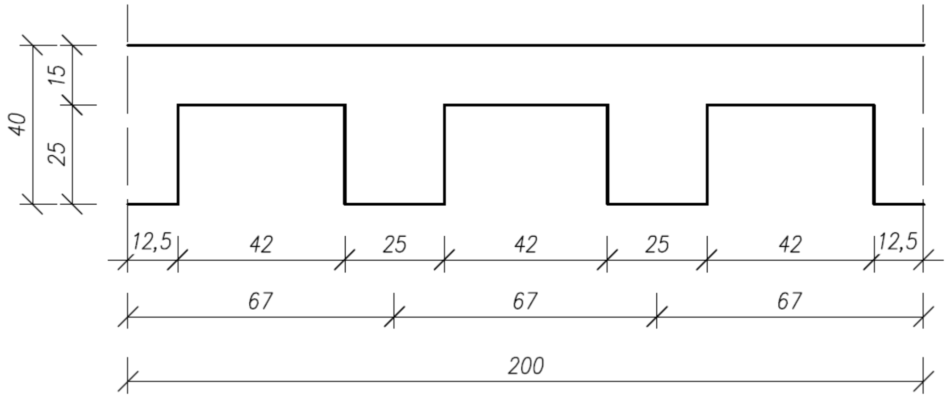

Fig.1. Cross-section of examined reinforced-concrete plate.

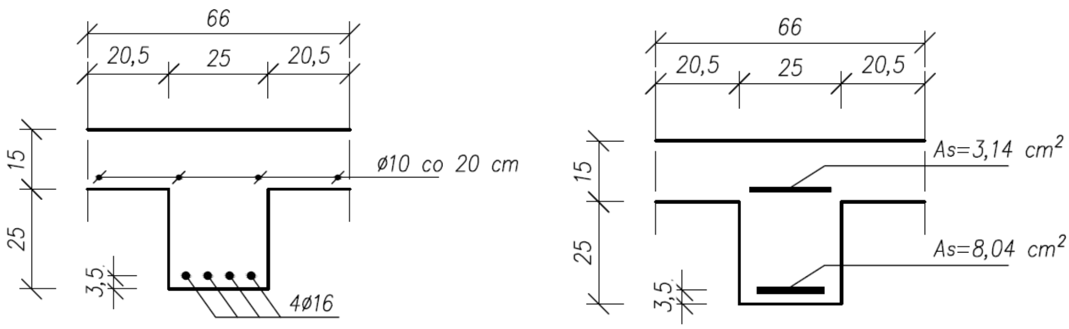

Fig.2. Steel-bar reinforcement details of repeatable plate element (left) and substitutional cross-section (right).

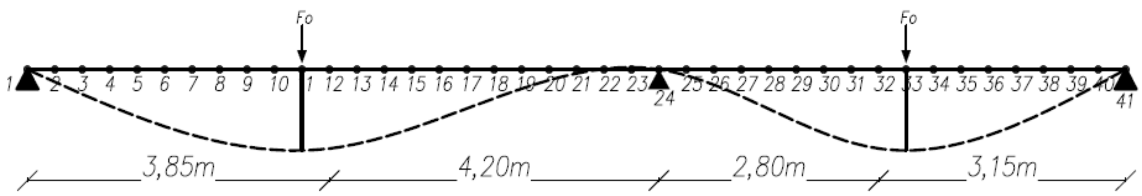

Fig.3. Static scheme of the analyzed structure with two devices generating dynamic forces in similar phase with the FEM discretization (and node numbers).

For the purpose of this exemplary case it was assumed that dynamic load is generated by unbalanced rotational motion of a device (industrial washing machine/spin-dryer Electrolux W5330N). which weights *307kg* net mass and generates a dynamic force of *4.6 kN* amplitude with frequency of *7.9Hz* (according to technical data sheet). The global mass matrix was properly modified in a way that it includes the mass of devices which are located at nodes as it is shown in Fig.4. The influence of two devices working in similar phase of motion is considered. Numerical calculations described below have been conducted assuming that the expected value of viscous damping coefficient equals $\zeta = 0.02$ (*2%* of critical damping). This is common practice in structural engineering of reinforced concrete structures; assumption of similar values of damping coefficients can be found in the literature. The following damping coefficients series has been taken into account: *0.0010*, *0.0025*, *0.0050*, *0.010*, *0.015*, *0.020*, *0.025*, *0.030*, *0.035*, *0.040* for this analysis purpose.

4. Description and calibration of the algorithm

The time step for the Runge-Kutta-Fehlberg procedure was dependent on the frequency of dynamic force and calculated in a way that each step was taken by one sixth of period of forced vibrations. Results of these calculations are described by a matrix in which there are vectors of displacement and velocity for each node of the beam (Fig.3). The node with greatest displacements was set as representative and taken into further consideration as it refers to Serviceability Limited State described in DIN4150. The vector of displacement and velocity for this point was exported into the file directory in 'Coma Separated Value' extension (.csv) and then imported into the second section of the programme. This series of actions was repeated for *10* various damping coefficients. Series of graphs presenting vibrations of the beam for *10* various damping coefficients within the range of *0* to *2* and *0* to *10* seconds of dynamic load action are presented below.

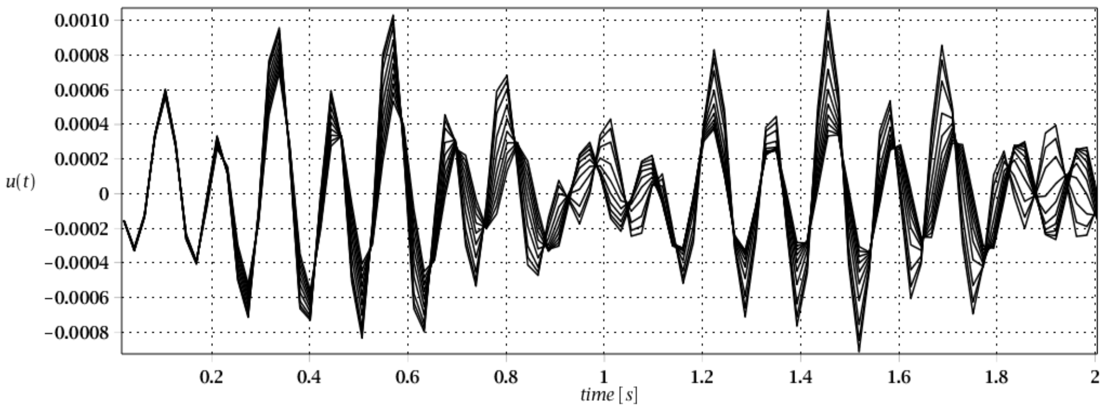

Fig.4. Development of maximum displacements for the first *2* seconds of structural vibration.

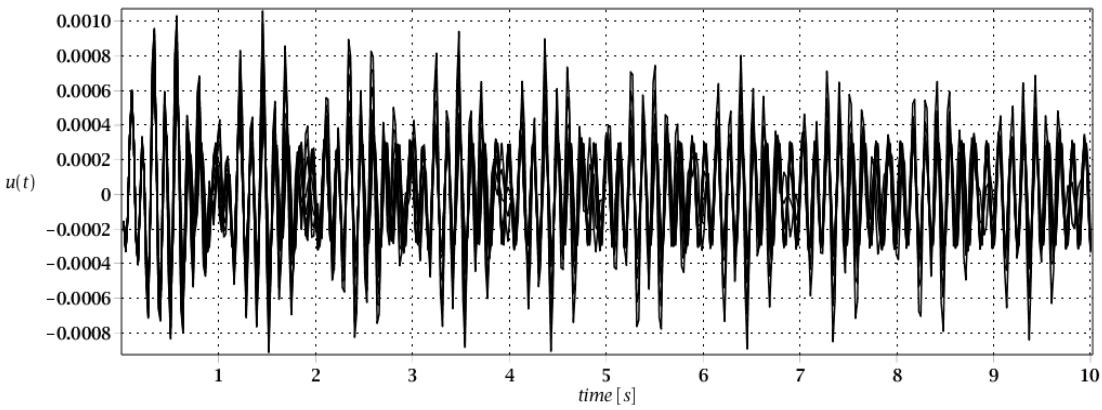

Fig.5. Development of maximum displacements for the first *10* seconds of structural vibration.

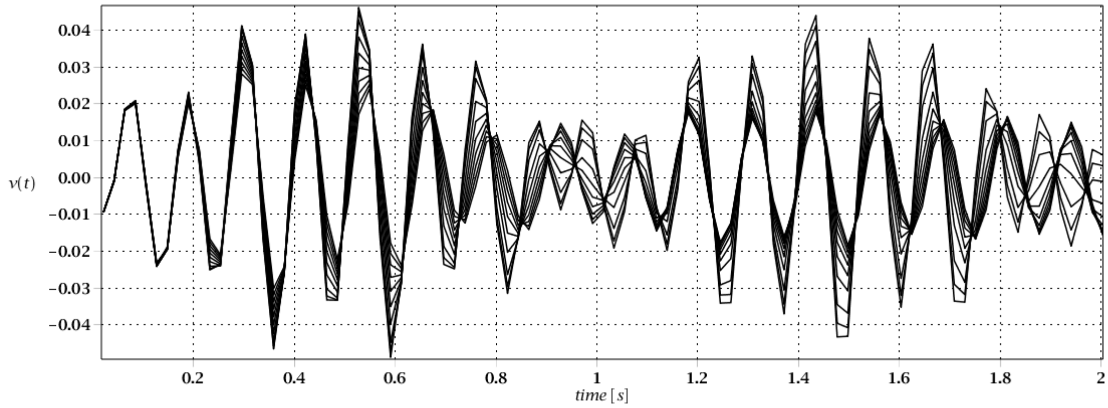

Fig.6. Development of maximum velocity for the first *2* seconds of structural vibration.

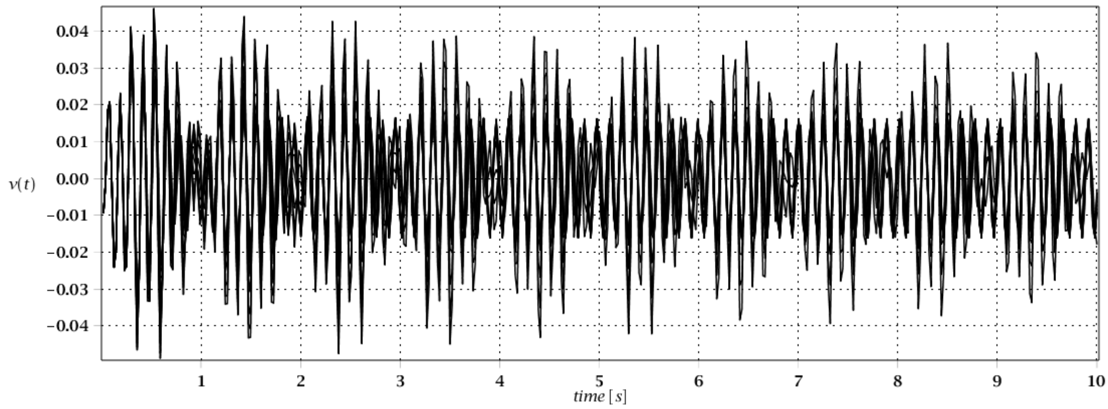

Fig.7. Development of maximum velocity for the first *10* seconds of structural vibration.

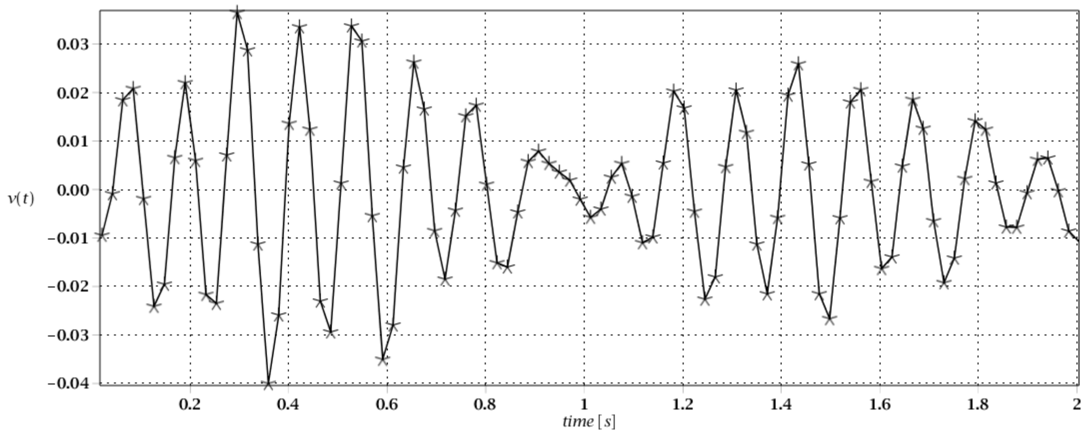

Fig.8. Distribution density of discrete values for damping coefficient equal to *0.02* for the first *2* seconds of motion.

In order to calibrate the time step of the Runge-Kutta-Fehlberg algorithm, there was conducted a comparison in which it was checked if discrete values of calculations steps were set densely enough to describe every peak value of oscillatory motion. It is shown in Figs 7-8 that discrete values calculated by 4^{th}

degree Runge-Kutta-Fehlberg technique (RKF45) in reference to continuous plot are determined not only in peak values but also along the oscillatory motion which confirms that the time step was set adequately.

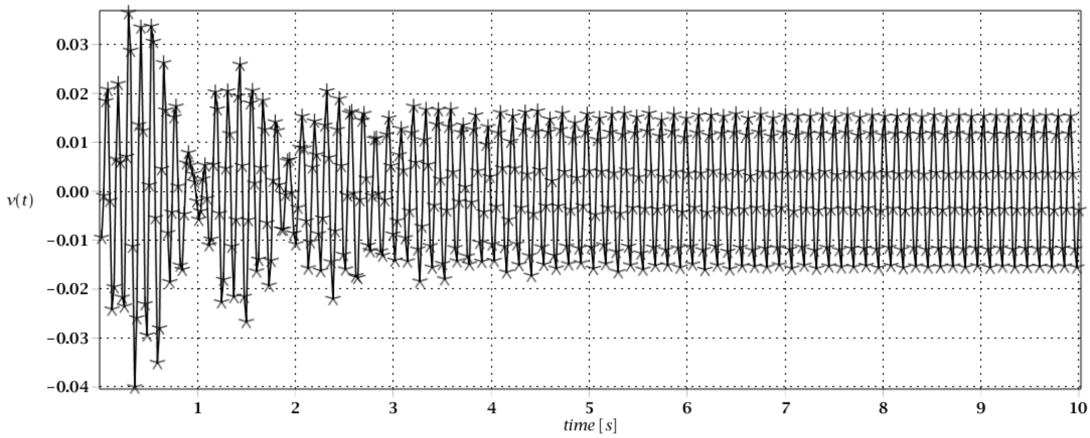

Fig.9. Distribution density of discrete values for damping coefficient equal to *0.02* for the first *10* seconds of motion.

The oscillatory equation was solved with *10* various damping coefficients which provides us with a possibility to build a structure-response function up to 10^{th} order polynomial. Having a set of displacements for each damping coefficient is necessary as the structure-response function is created for every time step. The structure-response function was created in a way that a damping coefficient is considered as a variable, time is considered as constant due to performing this procedure by every time step. The aforementioned function is created as a 10^{th} degree polynomial via the Least Squares Method (LSM) which was validated in terms of mean square error minimization.

At this point, random variable implementation takes place. The parameter treated as a variable in the structure-response function is now considered as a random variable. This approach can be used in reference to functions where there is more than one variable tested. The input coefficient of variation of the input random variable has been set as *0.1*.

The random variable is described not only by the expected value but also by its distribution. Every kind of distribution of the random variable, including symmetric and non-symmetric distribution, can be taken into account. Having those two parameters: the expected value and probability function that describes distribution of the random variable it is possible to perform an attempt to estimate the expected value of the previously built structure response function and so of the structure itself. Higher probabilistic moments are described and calculated as well giving as a result: the variance, standard deviation, coefficient of variation, skewness and kurtosis for the structure response function in every time step. These tests are conducted with the use of three methods: the Stochastic Perturbation Technique (SPT), Monte Carlo Simulation (MCS) and Semi-Analytical Method (SAM).

5. Discussion of the results

Detailed numerical results of computational analysis in reference to probabilistic figures of displacements and velocity of the structure during the first *10* seconds of vibration are presented below:

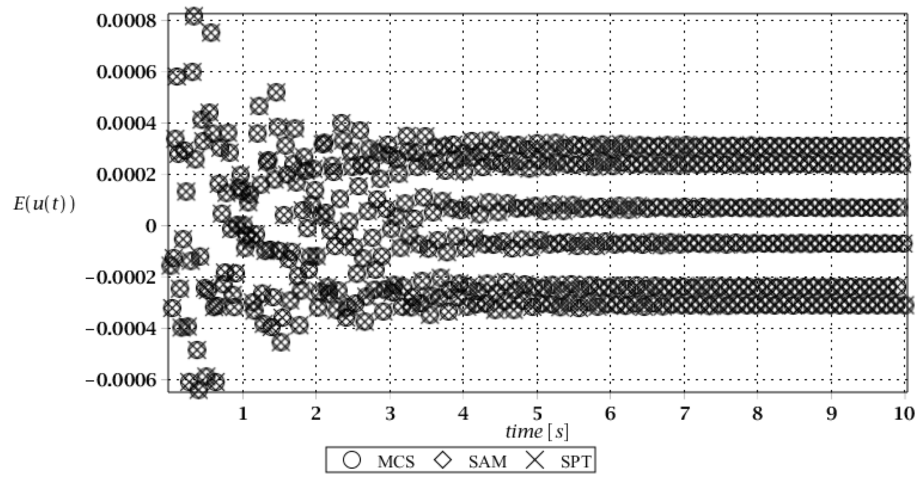

Fig.10. Expected values of displacement for the first *10* seconds of vibration analysis.

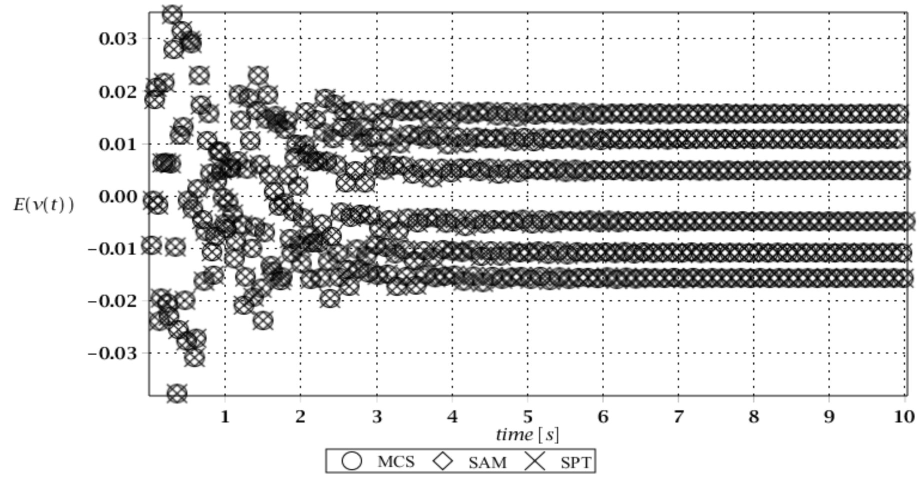

Fig.11. Expected values of velocity for the first *10* seconds of vibration analysis.

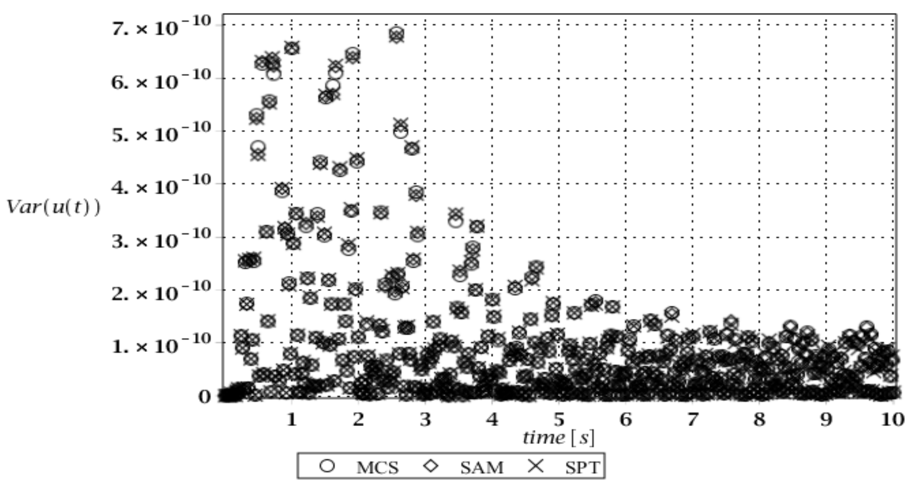

Fig.12. Variance of displacement for the first *10* seconds of vibration analysis.

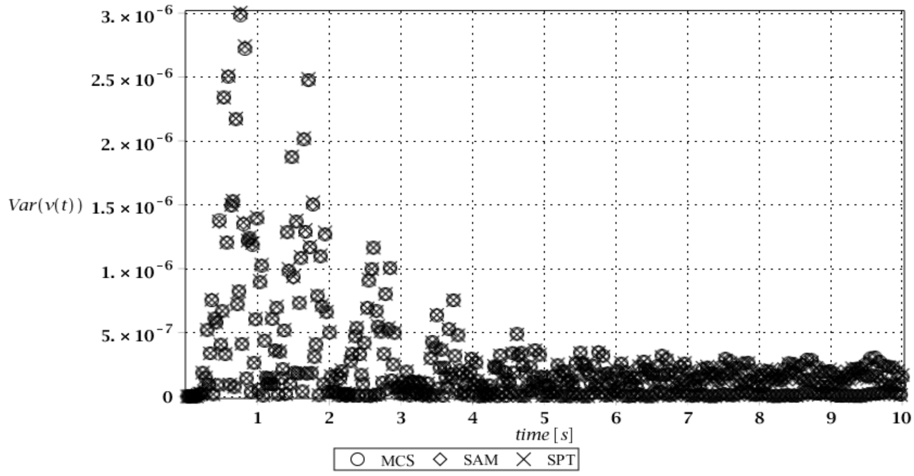

Fig.13. Variance of velocity for the first *10* seconds of vibration analysis.

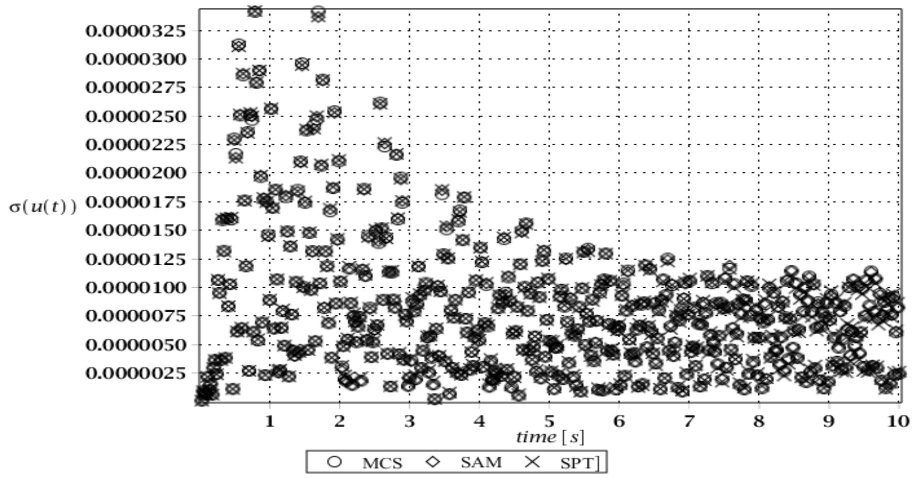

Fig.14. Standard deviation of displacement for the first *10* seconds of vibration analysis.

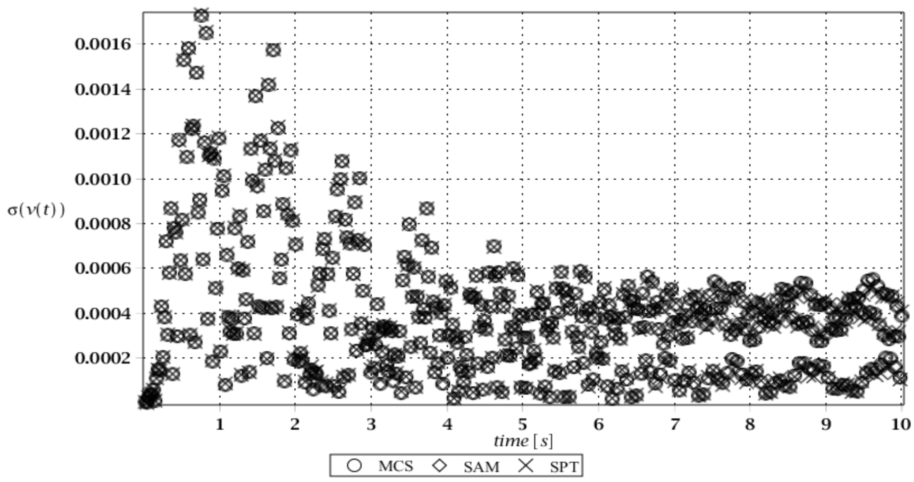

Fig.15. Standard deviation of velocity for the first *10* seconds of vibration analysis.

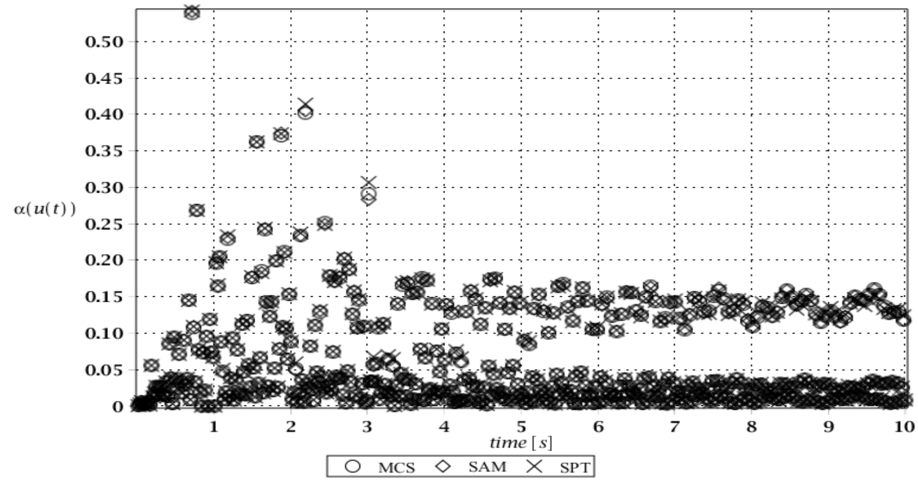

Fig.16. Coefficient of variation of displacement for the first *10* seconds of vibration analysis.

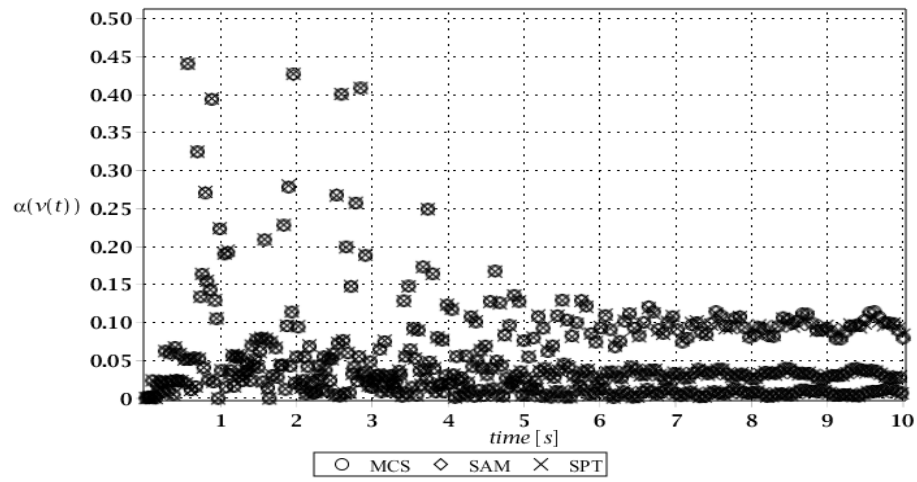

Fig.17. Coefficient of variation of velocity for the first *10* seconds of vibration analysis.

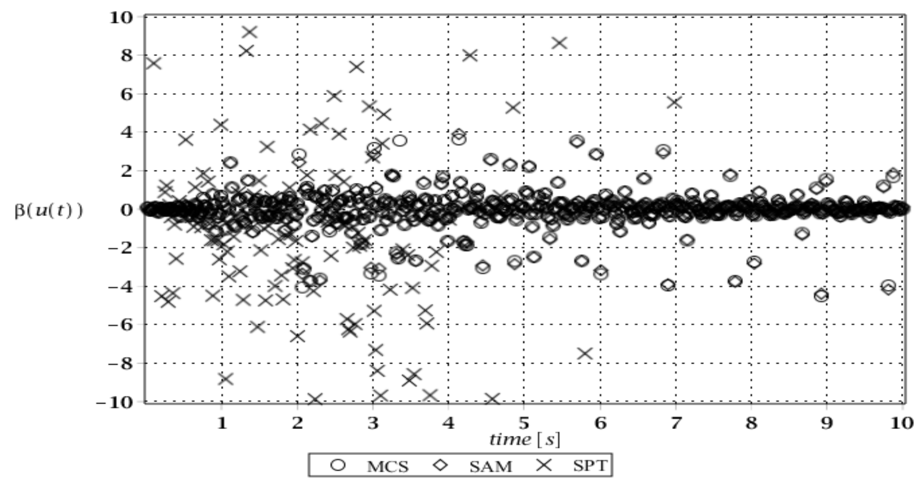

Fig.18. Skewness of displacement for the first *10* seconds of vibration analysis.

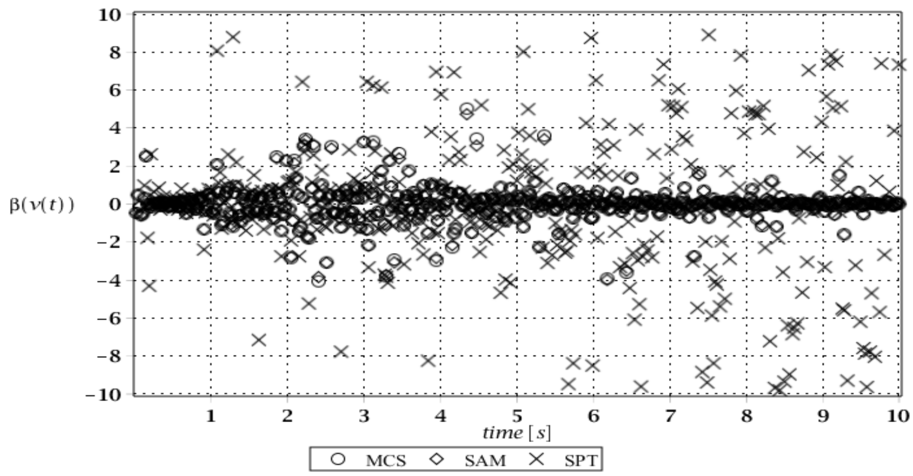

Fig.19. Skewness of velocity for the first *10* seconds of vibration analysis.

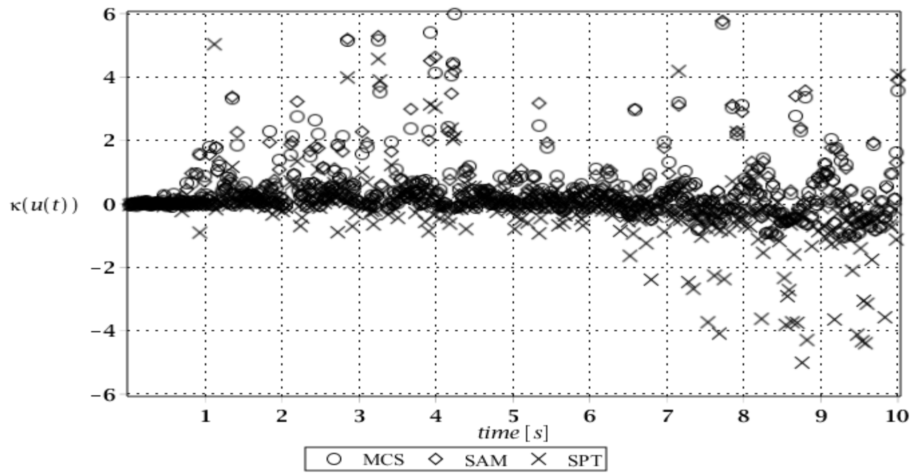

Fig.20. Kurtosis of displacement for the first *10* seconds of vibration analysis.

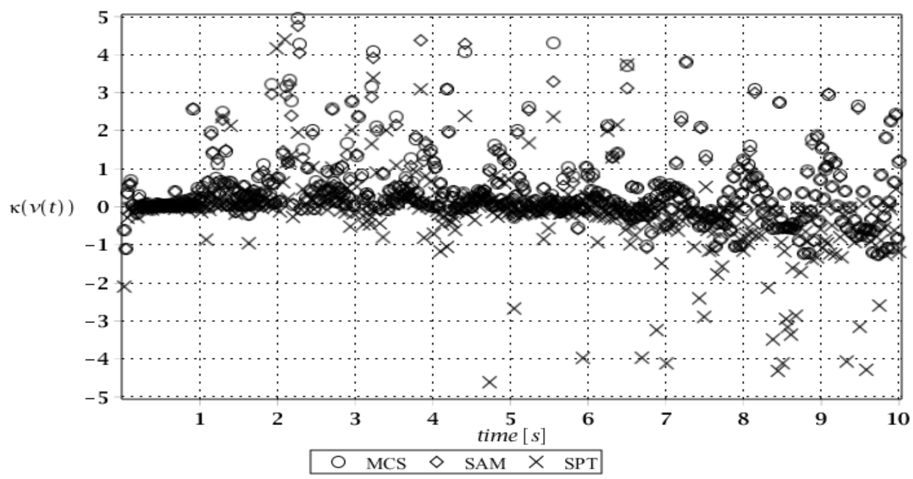

Fig.21. Kurtosis of displacement for the first *10* seconds of vibration analysis.

It is worth noting that numerical values of the coefficient of variation for the structure-response function exceed *0,10*. As it was mentioned before, the input value of the coefficient of variation associated with the damping coefficient was equal to *0.10*. The fact that the output value of COV in the structure-response function is frequently greater than the input value of COV of the damping coefficient means that progress of vibration with uncertain damping coefficient might be found problematic to predict until vibration stabilizes at certain magnitude. Time of stabilization is strictly dependent on the damping coefficient as it was proved by solving the governing equation repeatedly for *10* various damping coefficients, for which amplitudes of displacement stabilized earlier for every calculation performed within greater damping coefficient. What is more, values of kurtosis and skewness are not equal to zero. Few values of COV greater than *1.0* were rejected in further analysis as they were greater than the expected value itself and also described by nearly-zero displacements/velocities, so they would not be suspected to exceed figures of any limit state.

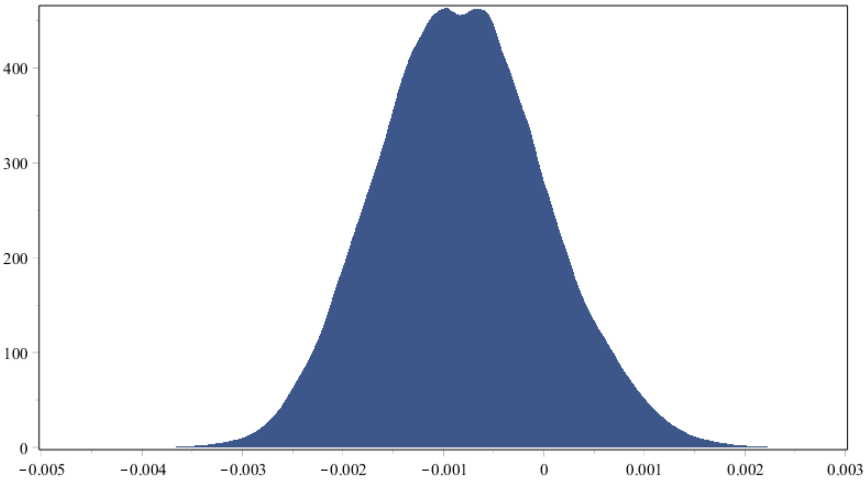

Fig.22. Histogram of the SRF obtained for the extreme coefficient of variation.

An SRF histogram of random samples distribution was set for the time step in which COV reached its peak value. This distribution was suspected not to be ideally symmetrical which was the case for the input value (damping coefficient). Regarding all the figures plotted above it can be said that results of expected values, variance and coefficient of variation among all three methods are very similar. From the perspective of the Stochastic Perturbation Technique, it can be said that estimation of expected values, standard deviations and coefficient of variation is very accurate. Probabilistic moments of higher order might be considered not accurate in comparison with Monte-Carlo simulations and the Semi-Analytical Method. Nevertheless, the SPT is accurate when it comes to reliability index estimation as expected value and variance are required only.

6. Calculation of the reliability index

For the purposes of reliability index estimation an assumption was made that the considered structure is classified as an industrial area. According to DIN4150, critical peak value of vibration velocity that guarantees structural integrity for given structure is equal to *40mm/sec*. This value of velocity is from now considered as Serviceability Limit State for a given case. Velocity of structure at every time-step at a chosen node is given directly due to implementation of the RFK algorithm in the Finite Element Method program for the governing equation of motion solution. This allows us to calculate directly the reliability index defined in [7].

The reliability index described in [4,21,22] is defined in a way that performance function g is distributed according to the Gaussian probability distribution. This reliability index is described by the mean value $E(g)$ and standard deviation $\sigma(g)$ of a function g as follows:

$$\beta = \frac{E(g)}{\sigma(g)}. \tag{6.1}$$

This performance function g is described as a difference of resistance R and effects of actions E, which means here a difference in-between admissible velocity V_{adm} and its extreme value determined in numerical model V_{ext}

$$g = R - E = V_{adm} - V_{ext}. \tag{6.2}$$

The reliability index may be transformed in the following way taking into account a complete lack of correlation in-between admissible and extreme velocities and the fact that the first one has definitely a deterministic character

$$\beta = \frac{E(V_{adm} - V_{ext})}{\sigma(V_{adm} - V_{ext})} = \frac{E(V_{adm}) - E(V_{ext})}{\sqrt{Var(V_{adm} - V_{ext})}} = \frac{E(V_{adm}) - E(V_{ext})}{\sigma(V_{ext})}. \tag{6.3}$$

Table 2. Reliability index calculation for the given data series.

No.	Time step No.	Time [s]	$E(V_{ext})$	$Var(V_{ext})$	$E(V_{adm})$	$Var(V_{adm})$	β [-]
Calculations performed in reference to velocity of structure							
1	446	*9.41*	$0.01574\, m \cdot s^{-1}$	$2.666 \cdot 10^{-9}\, m^2 \cdot s^{-2}$	$0.040\, m \cdot s^{-1}$	$0.00\, m^2 \cdot s^{-2}$	*469.85*

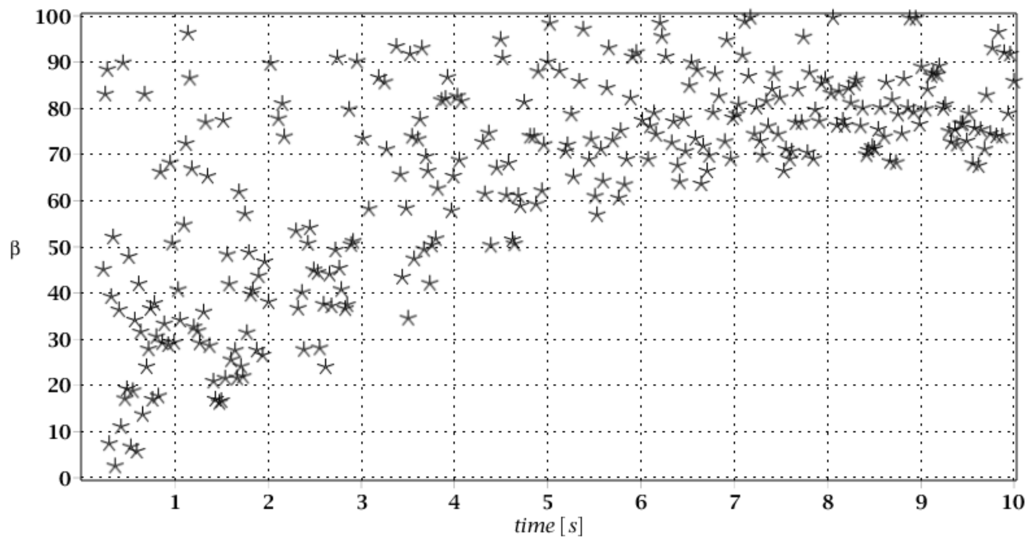

Fig.23. Reliability index time fluctuations.

Table 2 gives reliability index estimation provided on the basis of the numerical example completed above. This index has been calculated for such a discrete point in the beam FEM model, which corresponds to the largest displacement from the initial equilibrium position. Such a procedure has been repeated for any discrete time of structural vibrations under consideration, and furthermore, final graphs of the reliability index time fluctuations have been created using the first two moments of structural velocities resulting from the Stochastic Perturbation Method analysis. Numerical results presented above document very well that the largest uncertainty measure (and the smallest reliability measure at the same time) is associated with the very beginning of the vibration process, where random damping plays a decisive role. Then structural vibrations stabilize and this uncertainty becomes less and less important, however the reliability index exhibits some minimum value, which becomes permanent for the rest of vibration process.

7. Conclusions

The numerical algorithm and its computational implementation written in the system MAPLE makes it possible to determine stochastic vibrations of two-dimensional linear elastic multi-span beams with various kinematic boundary conditions (fixed or pin joints). Their bending stiffness can be estimated separately for each span or even for each finite element, which can be specifically implemented in order to simulate its loss for concrete beams resulting from some micro- and macro-cracks. The Finite Element Method procedure is based upon the 2-noded linear elements but its extension towards parabolic shape functions convenient to the 3-noded beam elements is quite straightforward; future application of the elastic support is taken into account, too. Dynamic loads applied to the beam can have harmonic nature or may be represented by any other time fluctuation. The algorithm might be transformed into FORTRAN or PYTHON programming language. The major challenge of this transformation could be to describe already built-in functionalities of Maple libraries for linear algebra, plotting tools and statistic package in those languages. The basic probabilistic characteristics of the structural responses such as expectations, variances, skewness and kurtosis have been obtained via initial determination of the polynomial responses in each node separately using the Least Squares Method and further application of the iterative generalized stochastic perturbation technique. Although the application presented above includes uncertainty in Rayleigh damping, further reliability analyses may include various design variables such as the Young modulus, corrosion process, geometrical imperfections or dynamic load characteristic. The Gaussian probability distribution has been chosen to model this damping coefficient according to the Central Limit Theorem. However, the stochastic perturbation technique presented here renders it possible to choose some other distributions functions also, which should be further investigated numerically.

It should be mentioned first of all that fundamental probabilistic characteristics for the concrete beam response computed according to the stochastic perturbation method coincide very well with these obtained using the Monte-Carlo simulation and semi-analytical technique also, which confirms the applicability of this technique for vibration analysis of linear elastic beams exhibiting uncertain damping; this effect is known from the previous numerical studies in probabilistic computational mechanics [23]. Calculations of probabilistic vibrations presented here demonstrate that time steps with the largest displacements (when the structure is about to move into opposite direction) correspond to the smallest values of the coefficient of variation. These maximum displacements are more likely to be dependent on stiffness and mass matrices rather than on the damping matrix. This assumption is strictly connected to the previous observation that a change in the input damping coefficient changes a period of time required to stabilize oscillatory motion rather than changing its maximum displacements after stabilization (at least for the case within damping coefficients smaller than the critical damping by an order of magnitude). From the engineering perspective it is worth noting that a quasi-static approach to solving dynamic problems might not guarantee a proper level of reliability. As it is shown in this paper, a proper time-depending analysis is recommended for since the reliability index decreases below a required value at certain time steps at least for structures subjected to dynamic load generated by industrial machines generating rotating vibrations.

Nomenclature

b – random damping parameter
$\boldsymbol{C}$ – damping matrix
C_{ki} – polynomial response function coefficients
E – effects of action upon the given structure
E – Young modulus of the structural material
$E(.)$ – expected value operator
f^0 – mean value of the function f
$\boldsymbol{F}$ – vector of nodal forces
g – limit function
h – time increment in dynamic analysis
J – inertia modulus of the given beam element
$\boldsymbol{K}$ – stiffness matrix
k_1, k_2, k_3, k_4 – RFK method parameters
$k^{(e)}$ – elemental stiffness matrix in the FEM
L – finite element length
$m^{(e)}$ – elemental mass matrix in the FEM
$\boldsymbol{M}$ – mass matrix
N – total number of random trials
p – stochastic perturbation method order
$p_b(x)$ – probability density function of random damping
R – structural resistance
t – time parameter in dynamic analysis
V_{adm} – admissible velocity
V_{ext} – extreme velocity
$Var(.)$ – variance operator
y – vector of nodal displacements
$\dot{y}$ – vector of nodal velocities
$\ddot{y}$ – vector of nodal accelerations
$\hat{\alpha},\hat{\beta}$ – Rayleigh coefficients
β – reliability index
ε – perturbation parameter
ρ – mass density

References

[1] Mayers A. (2009): *Vibration acceptance criteria.*– Australian Bulk Handling Review: issue March/April.

[2] DIN 4150 1-3:2001: *Vibrations in buildings.*

[3] AS 2670.1-201 Australian Standards: *Evaluation of human exposure to whole-body vibration, Part 1: General requirements.*

[4] Cornell C.A. (1968): *Engineering seismic risk analysis.*– Bulletin of Seismological Society of America, vol.58, No.5, pp.1583-1606.

[5] Madsen H.O., Krenk S. and Lind N.C. (1986): *Methods of Structural Safety.*– Prentice Hall, Englewood Cliffs.

[6] Melchers R.E. and Beck A.T. (2018): *Structural Reliability Analysis and Prediction.*– John Wiley & Sons, Hoboken, NJ, p.497.

[7] Valdebenito M.A., Jensen H.A., Schuëller G.I. and Caro F.E. (2012): *Reliability sensitivity estimation of linear systems under stochastic excitation.*– Computers and Structures, vol.92-93, pp.257-268.

[8] Soize C. (2013): *Stochastic modelling of uncertainties in computational structural dynamics – Recent theoretical advances.*– J. of Sound and Vibration, vol.332, No.10, pp.2379-2395.

[9] Roberts J.B. and Spanos P.D. (1990): *Random vibration and statistical linearization.*– Chichester, Wiley.

[10] Muscolino G., Ricciardi G. and Vasta M. (1997): *Stationary and non-stationary probability density function of non-linear oscillators.*– Int. J. of Non-Linear Mechanics, vol.32, No.6, pp.1051-1064.

[11] Roberts J.B. and Spanos P.D. (1986): *Stochastic averaging: an approximation method of solving random vibration problems.*– Int. J. Non-Linear Mechanics, vol.21, No.2, pp.111-134.

[12] Sobczyk K, Wędrychowicz S. and Spencer B.F. (1996): *Dynamics of structural systems with spatial randomness.*– Int. J. of Solids and Structures, vol.33, No.11, pp.1651-1669.

[13] Goller B., Pradlwarter H.J. and Schuëller G.I. (2013): *Reliability assessment in structural dynamics.*– Journal of Sound & Vibration, vol.332, pp.2488-2499.

[14] Kamiński M. (2013): *The Stochastic Perturbation Method for Computational Mechanics.*– Chichester, Wiley.

[15] Kamiński M. and Corigliano A. (2015): *Numerical solution of the Duffing equation with random coefficients.*– Meccanica, vol.50, pp.1841-1853.

[16] Hughes T.J.R. (2000): *The Finite Element Method – Linear Static and Dynamic Finite Element Analysis.– N*ew York, Dover Publications, Inc.

[17] Hutton D.V. (2004): *Fundamentals of Finite Element Analysis.*– McGraw-Hill.

[18] Han S.H., Benaroya H. and Wei T. (1999): *Dynamics of transversely vibrating beams using four engineering theories.*– Journal of Sound and Vibration, vol.225, No.5, pp.935-988.

[19] Liao S., Zhang Y. and Chen D. (2019) *Runge-Kutta Finite Element Method based on the characteristic for the incompressible Navier-Stokes equations.*– Advanced Applied Mathematics & Mechanics, vol.11, pp.1415-1435.

[20] Botasso C.L. (1997): *A new look at finite elements in time: a variational interpretation of Runge-Kutta methods.*– Applied Numerical Mathematics vol.25, pp.355-368.

[21] Eurocode 0 (2005)*: Basis of structural design. EN 1990:2002/A1.*– European Committee for Standardization, Brussels.

[22] Pradlwarter H.J. and Schuëller GI. (2010): *Uncertain linear systems in dynamics: Efficient stochastic reliability assessment.*– Computers and Structures, vol.88, pp.74-86.

[23] Kamiński M. (2015): *On the dual iterative stochastic perturbation-based finite element method in solid mechanics with Gaussian uncertainties.*– International Journal for Numerical Methods in Engineering, vol.104, No.11, pp.1038-1060.

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