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THERMAL STRESSES DUE TO NON-UNIFORM INTERNAL HEAT GENERATION IN FUNCTIONALLY GRADED HOLLOW CYLINDER

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Thermal stresses of a functionally graded hollow thick cylinder due to non-uniform internal heat generation are studied in this paper. Analytical solutions are obtained with radially varying properties by using the theory of elasticity. Thermal stresses distribution for different values of the powers of the module of elasticity and varying power law index of heat generation are studied. The results have been computed numerically and illustrated graphically.

Key words: functionally graded material, non-uniform heat source, thermal stresses, hollow thick cylinder, temperature.

1. Introduction

Functionally graded materials (FGM) are inhomogeneous materials, consisting of two or more different materials, engineered to have a continuously varying spatial composition profile. The FGM concept originated in Japan in 1984 during the space plane project. Materials with changing composition, microstructure or porosity across the volume of the material are referred to as the functionally graded materials.

Arefi *et al*. [1] presented a thermo-mechanical analysis of orthotropic rotating hollow structures of FGM due to thermo-mechanical loadings and derived temperature distribution, radial displacement and radial and circumferential stresses in general state. Ghanbari and Farhatnia [2] discussed the prediction of yielding onset and spread pattern in functionally graded thick-walled cylindrical vessel subjected to thermo-mechanical loading.

Jabbari *et al.* [3] performed a general analysis of one dimensional steady-state thermal stresses in a functionally graded hollow thick cylinder whereas Jabbari *et al.* [4] presented a general analysis of two-dimensional steady-state thermal stresses for a functionally graded hollow thick cylinder. Kedar and Deshmukh [5] presented an analysis of a thin clamped hollow disk under unsteady temperature field due to point heat source in the form of thermal stresses.

Nayak and Mondal [6] analyzed a functionally graded thick cylindrical vessel with radially varying properties and obtained stresses with the consideration that the properties of the material vary with the power law of radius. Obata and Noda [7] considered steady thermal stresses in a functional graded hollow circular cylinder and hollow sphere and discussed the influence of the inside radius size on stresses. Pawar *et al.* [8] determined thermal stresses in FGM hollow sphere due to non-uniform internal heat generation with radially varying properties by using theory of elasticity and Pawar *et al.* [9] presented the thermoelastic analysis of FGM solid sphere subjected to non-uniform heat source under the constant surface temperature. Yildirim [10] presented a thermo-mechanical analysis of sphere made of non-homogeneous isotropic materials and proposed the closed form formulas for the elastic fields in a simple-power-law graded spheres subjected to steady-state thermal and internal/external pressure loads.

Zimmerman and Lutz [11] derived the exact solution for the problem of uniformly heating cylinder and showed that the radial and tangential stresses are largest in magnitude at the center whereas the deviotric

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stress is largest in magnitude at the outer edge of the cylinder. In the present paper, we have studied the thermal stresses based on the theory of uncoupled thermoelasticity in a functionally graded hollow thick-cylinder subjected to non-uniform internal heat generation and discussed the thermal stresses distribution for different values of the powers of the module of elasticity and varying power law index of heat generation. We obtained the expressions for radial and circumferential thermal stresses by using the theory of elasticity. The variation of thermal stresses is also shown graphically.

2. Formulation of problem

Consider an FGM hollow cylinder with the inner radius 'a' and outer radius 'b', temperature T_a at the inner surface and T_b at the outer surface with non-uniform heat generation 'q' within the solid.

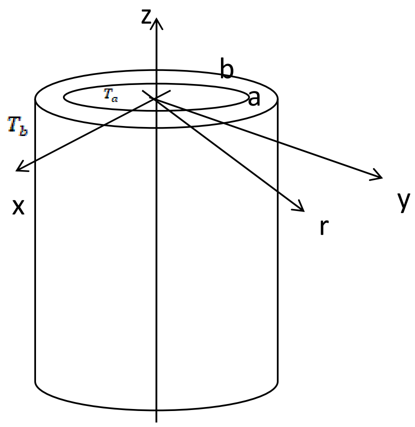


Fig.1. FGM hollow cylinder.

The properties in cylindrical co-ordinates z and θ are identical. The material properties are assumed to vary as power function in radial direction. The cylinder is graded in the radial direction so that the material properties of elasticity modulus, thermal expansion coefficient and thermal conductivity are functions of r only. Non-uniform heat is generated within it and is also a function of r. The following power law functions of radius in the radial direction are considered as in Ghanbari and Farhatnia [2]:

- elasticity modulus:

$$E = E_0 r^{m_1} ,$$

- coefficient of thermal expansion:

$$\alpha = \alpha_0 r^{m_2} , \tag{2.1}$$

- thermal conductivity:

$$k = k_0 r^{m_3} ,$$

- non-uniform heat generation:

$$q = q_0 r^{m_4}$$

where m_1, m_2, m_3 and m_4 are parameters. E_0, α_0, k_0 and q_0 are reference values of the Young modulus, thermal expansion coefficient, thermal conductivity and non-uniform heat generation, respectively.

In the steady state condition, in cylindrical co-ordinates and under first kind thermal boundary conditions as in Obata and Noda [7] with non-uniform heat generation, the heat conduction equation may be obtained as follows:

$$\frac{1}{r}\frac{d}{dr}\left[rk_0 r^{m_3}\frac{dT}{dr}\right]+q_0 r^{m_4}=0 \qquad a\le r\le b, \qquad t>0. \tag{2.2}$$

Boundary conditions are

$$T=T_a \qquad \text{at} \qquad r=a, \tag{2.3}$$

$$T=T_b \qquad \text{at} \qquad r=b. \tag{2.4}$$

3. Thermoelastic solution

The properties in the cylindrical co-ordinates z and θ directions are identical and u is the displacement component in the radial direction. The non-zero strain components are

$$\varepsilon_{rr}=\frac{du}{dr}, \qquad \varepsilon_{\theta\theta}=\frac{u}{r}. \tag{3.1}$$

The stress components σ_{rr} and $\sigma_{\theta\theta}$ in the radial and circumferential direction in terms of the displacement component u, for plane strain conditions are given by

$$\sigma_{rr}=\frac{E}{(1+\sigma)(1-2\sigma)}[(1-\sigma)\frac{du}{dr}+\sigma\frac{u}{r}-(1+\sigma)\alpha_t T], \tag{3.2}$$

$$\sigma_{\theta\theta}=\frac{E}{(1+\sigma)(1-2\sigma)}[\sigma\frac{du}{dr}+(1-\sigma)\frac{u}{r}-(1+\sigma)\alpha_t T] \tag{3.3}$$

where ε_{rr} and $\varepsilon_{\theta\theta}$ are strains in the radial and circumferential directions, T is the temperature change determined form the heat conduction equation (2.2), α_t is the coefficient of thermal expansion.

The equilibrium equation in the radial direction in the absence of body force and inertia term is given as below:

$$r\frac{d\sigma_{rr}}{dr}+\left(\sigma_{rr}-\sigma_{\theta\theta}\right)=0. \tag{3.4}$$

The cylinder is subjected to the traction free boundary conditions, i.e.

$$\sigma_{rr}=0 \qquad \text{at} \qquad r=a, \tag{3.5}$$

and

$$\sigma_{\theta\theta}=0 \qquad \text{at} \qquad r=b. \tag{3.6}$$

Therefore, Eqs (2.1)-(2.4) and (3.1)-(3.6) constitute the mathematical formulation of the problem.

4. Temperature distribution function

The solution of heat conduction Eq.(2.2) is obtained as:

$$T(r)=Q_1 r^{m_4-m_3+2}+C_1 r^{-m_3}+C_2 \qquad \text{where} \qquad m_3\neq 0 \tag{4.1}$$

and

$$Q_1=\frac{-q_0}{k_0\left(m_4+2\right)\left(m_4-m_3+2\right)}, \tag{4.2}$$

$$C_1=-\frac{B_1}{k_0 m_3}, \qquad m_3\neq 0 . \tag{4.3}$$

The constant of integration C_1 and C_2 can be determined by using boundary conditions (2.3)-(2.4) in Eq.(4.1)

$$C_1=\frac{\left(T_a-T_b\right)+Q_1\left(b^{m_4-m_3+2}-a^{m_4-m_3+2}\right)}{b^{-m_3}-a^{-m_3}}, \tag{4.4}$$

$$C_2=T_b+\frac{\left(T_a-T_b\right)b^{-m_3}}{\left(b^{-m_3}-a^{-m_3}\right)}-Q_1\left\{\frac{b^{m_4-m_3+2}-\left(b^{m_4-m_3+2}-a^{m_4-m_3+2}\right)b^{-m_3}}{\left(b^{-m_3}-a^{-m_3}\right)}\right\}. \tag{4.5}$$

The parameters m_3 and m_4 are chosen such that the denominator is non-zero, the temperature distribution function is obtained as:

$$T\left(r\right)=T_b+\frac{\left(T_a-T_b\right)\left(b^{-m_3}-r^{-m_3}\right)}{\left(b^{-m_3}-a^{-m_3}\right)}+Q_1\left\{\left(r^{m_4-m_3+2}-b^{m_4-m_3+2}\right)+\right.$$
$$\left.+\frac{\left(b^{m_4-m_3+2}-a^{m_4-m_3+2}\right)\left(b^{-m_3}-r^{-m_3}\right)}{\left(b^{-m_3}-a^{-m_3}\right)}\right\}, \tag{4.6}$$

for

$$R=P\left\{\frac{\left(\sigma m_1+\sigma-1\right)}{\left(1-\sigma\right)}\right\},$$

$$T\left(r\right)=Q_2 r^{m_4+2}+C_3\ln r+C_4 \tag{4.7}$$

where

$$Q_2=\frac{-q_0}{k_0\left(m_4+2\right)^2}, \tag{4.8}$$

$$C_3=\frac{B_2}{k_0}. \tag{4.9}$$

Using the boundary conditions (2.3)-(2.4) in (4.7), we obtain the constants

$$C_3 = \frac{(T_b - T_a) - Q_2\left(b^{m_4+2} - a^{m_4+2}\right)}{\ln\frac{b}{a}}, \tag{4.10}$$

$$C_4 = T_b - \frac{(T_b - T_a)\ln b}{\ln\frac{b}{a}} - Q_2\left\{b^{m_4+2} - \frac{\left(b^{m_4+2} - a^{m_4+2}\right)\ln b}{\ln\frac{b}{a}}\right\} \tag{4.11}$$

and the temperature distribution function is given by

$$T(r) = T_b - \frac{(T_b - T_a)\ln\frac{b}{r}}{\ln\frac{b}{a}} + Q_2\left\{\frac{\left(b^{m_4+2} - a^{m_4+2}\right)\ln\frac{b}{r}}{\ln\frac{b}{a}} + \left(r^{m_4+2} - b^{m_4+2}\right)\right\}. \tag{4.12}$$

5. Thermoelastic stresses

The equilibrium Eq.(3.4) in terms of displacement component u on using the functional relation (2.1) and Eqs (3.2)-(3.3).

$$\Pr^2 \frac{d^2u}{dr^2} + Qr\frac{du}{dr} + Ru = L_1 r^{m_2+m_4-m_3+3} + M_1 r^{m_2-m_3+1} + N_1 r^{m_2+1} \qquad \text{for} \qquad m_3 \neq 0, \tag{5.1}$$

$$\Pr^2 \frac{d^2u}{dr^2} + Qr\frac{du}{dr} + Ru = L_2 r^{m_2+m_4+3} + M_2 r^{m_2+1}\ln r + N_2 r^{m_2+1} \qquad \text{for} \qquad m_3 = 0 \tag{5.2}$$

where

$$P = \frac{(1-\sigma)}{(1+\sigma)}, \qquad Q = P(m_1+1), \qquad R = P\left\{\frac{(\sigma m_1 + \sigma - 1)}{(1-\sigma)}\right\}, \tag{5.3}$$

$$L_1 = \alpha_{t0} Q_1 (m_1 + m_2 + m_4 - m_3 + 2), \tag{5.4}$$

$$M_1 = \alpha_{t0} C_1 (m_1 + m_2 - m_3), \tag{5.5}$$

$$N_1 = \alpha_{t0} C_2 (m_1 + m_2), \tag{5.6}$$

$$L_2 = \alpha_{t0} Q_2 (m_1 + m_2 + m_4 + 2), \tag{5.7}$$

$$M_2 = \alpha_{t0} C_3 (m_1 + m_2), \tag{5.8}$$

$$N_2 = \alpha_{t0}\left\{C_3 + C_4(m_1 + m_2)\right\}. \tag{5.9}$$

The general solutions of Eqs (5.1)-(5.2) are obtained by adding particular solutions to the complimentary solution of homogenous form. The complimentary function u_c is

$$u_c(r) = Xr^s \tag{5.10}$$

where X is constant.
Using Eq.(5.10) in homogeneous form of Eq.(5.1) or Eq.(5.2), we get

$$\Pr^2 \frac{d^2}{dr^2}\left(Xr^s\right) + Qr\frac{d}{dr}\left(Xr^s\right) + R\left(Xr^s\right) = 0$$

or

$$\Pr^2 Xs(s-1)r^{s-2} + QrXsr^{s-1} + RXr^s = 0\,,$$

on simplification, we get

$$Ps^2 + (Q-P)s + R = 0\,, \tag{5.11}$$

Equation (5.11) has two roots

$$s_{1,2} = \frac{(P-Q) \pm \sqrt{(Q-P)^2 - 4PR}}{2P}\,. \tag{5.12}$$

Thus the complimentary functions for Eqs (5.1) and (5.2) are

$$u_c(r) = X_1 r^{s_1} + X_2 r^{s_2} \qquad \text{for} \qquad m_3 \neq 0 \tag{5.13}$$

and

$$u_c(r) = X_3 r^{s_1} + X_4 r^{s_2} \qquad \text{for} \qquad m_3 = 0\,. \tag{5.14}$$

The particular solutions for Eqs (5.1) and (5.2) are considered as

$$u_p(r) = Y_1 r^{m_2+m_4-m_3+3} + Y_2 r^{m_2-m_3+1} + Y_3 r^{m_2+1} \qquad \text{for} \qquad m_3 \neq 0 \tag{5.15}$$

and

$$u_p(r) = Y_4 r^{m_2+m_4+3} + \left(Y_5 \ln r + Y_6\right) r^{m_2+1} \qquad \text{for} \qquad m_3 = 0\,. \tag{5.16}$$

Substituting Eq.(5.15) in Eq.(5.1) and Eq.(5.16) in Eq.(5.2) and equating the coefficients of identical powers using the values of L_1, M_1, N_1, L_2, M_2 and N_2 from Eqs (5.4)-(5.9), we obtain

$$Y_1 = \frac{\alpha_{t0} Q_1 \left(m_1 + m_2 + m_4 - m_3 + 2\right)}{\left(m_2 + m_4 - m_3 + 3\right)\left[P\left(m_2 + m_4 - m_3 + 2\right) + Q\right] + R}\,, \tag{5.17}$$

$$Y_2 = \frac{\alpha_{t0} C_1 \left(m_1 + m_2 - m_3\right)}{\left(m_2 - m_3 + 1\right)\left[P\left(m_2 - m_3\right) + Q\right] + R}\,, \qquad C_1 = -\frac{B_1}{k_0 n_3}\,, \tag{5.18}$$

$$Y_3 = \frac{\alpha_{t_0} C_2 (m_1 + m_2)}{(m_2 + 1)[Pm_2 + Q] + R}, \tag{5.19}$$

$$Y_4 = \frac{\alpha_{t_0} Q_2 (m_1 + m_2 + m_4 + 2)}{(m_2 + m_4 + 3)[P(m_2 + m_4 + 2) + Q] + R}, \tag{5.20}$$

$$Y_5 = \frac{\alpha_{t_0} C_3 (m_1 + m_2)}{(m_2 + 1)[Pm_2 + Q] + R}, \qquad C_3 = \frac{B_2}{k_0}, \tag{5.21}$$

$$\begin{aligned} Y_6 = &\frac{1}{[(m_2+1)(Pm_2+Q)+R]^2}\left\{\alpha_{t_0}[C_3 + C_4(m_1+m_2)][(m_2+1)(Pm_2+Q)+R]+\right. \\ &\left.-\alpha_{t_0} C_3 (m_1+m_2)[P(2m_2+1)+Q]\right\}. \end{aligned} \tag{5.22}$$

The general solution $u(r)$ of Eq.(5.1) is obtained for $m_3 \neq 0$ as

$$u(r) = X_1 r^{s_1} + X_2 r^{s_2} + Y_1 r^{m_2+m_4-m_3+3} + Y_2 r^{m_2-m_3+1} + Y_3 r^{m_2+1} \tag{5.23}$$

and general solution $u(r)$ of Eq.(5.2) is obtained for $m_3 = 0$ as

$$u(r) = X_3 r^{s_1} + X_4 r^{s_2} + Y_4 r^{m_2+m_4+3} + (Y_5 \ln r + Y_6) r^{m_2+1}, \tag{5.24}$$

using Eq.(5.23) in Eqs (3.2)-(3.3), we get:

- for $m_3 \neq 0$

$$\begin{aligned} \sigma_{rr} = &\frac{E_0}{(1+\sigma)(1-2\sigma)}\Big[\{(1-\sigma)s_1 + \sigma\} X_1 r^{m_1+s_1-1} + \{(1-\sigma)s_2 + \sigma\} X_2 r^{m_1+s_2-1} + \\ &+\big[\{(1-\sigma)(m_2+m_4-m_3+3)+\sigma\} Y_1 - (1+\sigma)\alpha_{t_0} Q_1\big] r^{m_1+m_2+m_4-m_3+2} + \\ &+\big[\{(1-\sigma)(m_2-m_3+1)+\sigma\} Y_2 - (1+\sigma)\alpha_{t_0} C_1\big] r^{m_1+m_2-m_3} + \\ &+\big[\{(1-\sigma)(m_2+1)+\sigma\} Y_3 - (1+\sigma)\alpha_{t_0} C_2\big] r^{m_1+m_2}\Big], \end{aligned} \tag{5.25}$$

$$\begin{aligned} \sigma_{\theta\theta} = &\frac{E_0}{(1+\sigma)(1-2\sigma)}\Big[\{(1-\sigma) + \sigma s_1\} X_1 r^{m_1+s_1-1} + \{(1-\sigma) + \sigma s_2\} X_2 r^{m_1+s_2-1} + \\ &+\big[\{\sigma(m_2+m_4-m_3+3)+(1-\sigma)\} Y_1 - (1+\sigma)\alpha_{t_0} Q_1\big] r^{m_1+m_2+m_4-m_3+2} + \\ &+\big[\{\sigma(m_2-m_3+1)+(1-\sigma)\} Y_2 - (1+\sigma)\alpha_{t_0} C_1\big] r^{m_1+m_2-m_3} + \\ &+\big[\{\sigma(m_2+1)+(1-\sigma)\} Y_3 - (1+\sigma)\alpha_{t_0} C_2\big] r^{m_1+m_2}\Big]. \end{aligned} \tag{5.26}$$

Now using Eq.(5.24) in Eqs (3.2)-(3.3), we get:

- for $m_3 = 0$

$$\begin{aligned}\sigma_{rr} = &\frac{E_0}{(1+\sigma)(1-2\sigma)}\Big[\{(1-\sigma)s_1+\sigma\}X_3 r^{m_1+s_1-1}+\{(1-\sigma)s_2+\sigma\}X_4 r^{m_1+s_2-1}+\\ &+\Big[\{(1-\sigma)(m_2+m_4)+(3-2\sigma)\}Y_4-(1+\sigma)\alpha_{t_0}Q_2]\Big]r^{m_1+m_2+m_4+2}+\\ &+\Big[\{(1-\sigma)m_2+1\}Y_5-(1+\sigma)\alpha_{t_0}C_3]\Big]r^{m_1+m_2}\ln r+\\ &+\Big[\{(1-\sigma)m_2+1\}Y_6+(1-\sigma)Y_5-(1+\sigma)\alpha_{t_0}C_4\Big]r^{m_1+m_2}\Big],\end{aligned} \tag{5.27}$$

$$\begin{aligned}\sigma_{\theta\theta} = &\frac{E_0}{((1+\sigma))((1-2\sigma))}\Big[\{(1-\sigma)+\sigma s_1\}X_3 r^{m_1+s_1-1}+\{(1-\sigma)+\sigma s_2\}X_4 r^{m_1+s_2-1}+\\ &+\Big[\{\sigma(m_2+m_4+3)+(1-\sigma)\}Y_4-(1+\sigma)\alpha_{t_0}Q_2\Big]r^{m_1+m_2+m_4+2}+\\ &+\Big[\{\sigma(m_2+1)+(1-\sigma)\}Y_5-(1+\sigma)\alpha_{t_0}C_3]\Big]r^{m_1+m_2}\ln r+\\ &+\Big[\sigma Y_5+\{\sigma((m_2+1)+(1-\sigma)\}Y_6-(1+\sigma)\alpha_{t_0}C_4\Big]r^{m_1+m_2}\Big]\end{aligned} \tag{5.28}$$

where X_1, X_2, X_3 and X_4 are the constants obtained by using the conditions (3.5) and (3.6). Then the expressions for thermal stresses are obtained from Eqs (5.25)-(5.28):

- for $m_3 \neq 0$

$$\begin{aligned}\sigma_{rr} = &\frac{E_0\alpha_{t_0}}{A_1}\Big[\!\Big[Q_1\Big\{Z_1\Big[\Big(a^{m_1+s_2-1}b^{m_1+m_2+m_4-m_3+2}+\\ &-a^{m_1+m_2+m_4-m_3+2}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\Big(b^{m_1+s_1-1}a^{m_1+m_2+m_4-m_3+2}+\\ &-b^{m_1+m_2+m_4-m_3+2}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}+A_1 r^{m_1+m_2+m_4-m_3+2}\Big]+\\ &-A_2Z_2\Big[\Big(a^{m_1+s_2-1}b^{m_1+m_2-m_3}-a^{m_1+m_2-m_3}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\\ &+\Big(b^{m_1+s_1-1}a^{m_1+m_2-m_3}-b^{m_1+m_2-m_3}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}+A_1 r^{m_1+m_2-m_3}\Big]+\\ &-A_3Z_3\Big[\Big(a^{m_1+s_2-1}b^{m_1+m_2}-a^{m_1+m_2}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\\ &+\Big(b^{m_1+s_1-1}a^{m_1+m_2}-b^{m_1+m_2}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}+A_1 r^{m_1+m_2}\Big]\Big\}+\\ &-A_4Z_2\Big[\Big(a^{m_1+s_2-1}b^{m_1+m_2-m_3}-a^{m_1+m_2-m_3}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\\ &+\Big(b^{m_1+s_1-1}a^{m_1+m_2-m_3}-b^{m_1+m_2-m_3}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}+A_1 r^{m_1+m_2-m_3}\Big]+\\ &+A_5Z_3\Big[\Big(a^{m_1+s_2-1}b^{m_1+m_2}-a^{m_1+m_2}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\\ &+\Big(b^{m_1+s_1-1}a^{m_1+m_2}-b^{m_1+m_2}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}+A_1 r^{m_1+m_2}\Big]\Big]\!\Big],\end{aligned} \tag{5.29}$$

$$\sigma_{\theta\theta}=\frac{E_0\alpha_{t_0}}{A_1}\Bigg[\!\!\Bigg[Q_1\Bigg\{Z_1\Bigg[\frac{(\sigma s_1+1-\sigma)}{[(1-\sigma)s_1+\sigma]}\Big(a^{m_1+s_2-1}b^{m_1+m_2+m_4-m_3+2}+$$
$$-a^{m_1+m_2+m_4-m_3+2}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\frac{(\sigma s_2+1-\sigma)}{[(1-\sigma)s_2+\sigma]}\times$$
$$\times\Big(b^{m_1+s_1-1}a^{m_1+m_2+m_4-m_3+2}-b^{m_1+m_2+m_4-m_3+2}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}\Bigg]+$$
$$+Z_4A_1r^{m_1+m_2+m_4-m_3+2}-A_2Z_2\Bigg[\frac{(\sigma s_1+1-\sigma)}{[(1-\sigma)s_1+\sigma]}\Big(a^{m_1+s_2-1}b^{m_1+m_2-m_3}+$$
$$-a^{m_1+m_2-m_3}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+\frac{(\sigma s_2+1-\sigma)}{[(1-\sigma)s_2+\sigma]}\Big(b^{m_1+s_1-1}a^{m_1+m_2-m_3}+$$
$$-b^{m_1+m_2-m_3}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}-Z_5A_1A_2r^{m_1+m_2-m_3}-Z_6A_1A_3r^{m_1+m_2}+$$
$$-A_3Z_3\Bigg[\frac{(\sigma s_1+1-\sigma)}{[(1-\sigma)s_1+\sigma]}\Big(a^{m_1+s_2-1}b^{m_1+m_2}-a^{m_1+m_2}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+ \tag{5.30}$$
$$+\frac{(\sigma s_2+1-\sigma)}{[(1-\sigma)s_2+\sigma]}\Big(b^{m_1+s_1-1}a^{m_1+m_2}-b^{m_1+m_2}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}\Bigg]\Bigg]\Bigg\}+$$
$$-A_4Z_2\Bigg[\frac{(\sigma s_1+1-\sigma)}{[(1-\sigma))s_1+\sigma]}\Big(a^{m_1+s_2-1}b^{m_1+m_2-m_3}-a^{m_1+m_2-m_3}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+$$
$$+\frac{(\sigma s_2+1-\sigma)}{[(1-\sigma)s_2+\sigma]}\Big(b^{m_1+s_1-1}a^{m_1+m_2-m_3}-b^{m_1+m_2-m_3}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}\Bigg]+$$
$$-Z_5A_1A_4r^{m_1+m_2-m_3}+Z_6A_1A_5r^{m_1+m_2}+$$
$$+A_5Z_3\Bigg[\frac{(\sigma s_1+1-\sigma)}{[(1-\sigma)s_1+\sigma]}\Big(a^{m_1+s_2-1}b^{m_1+m_2}-a^{m_1+m_2}b^{m_1+s_2-1}\Big)r^{m_1+s_1-1}+$$
$$+\frac{(\sigma s_2+1-\sigma)}{[(1-\sigma)s_2+\sigma]}\Big(b^{m_1+s_1-1}a^{m_1+m_2}-b^{m_1+m_2}a^{m_1+s_1-1}\Big)r^{m_1+s_2-1}\Bigg]\Bigg]\!\!\Bigg]$$

where

$$A_1=a^{m_1+s_1-1}b^{m_1+s_2-1}-a^{m_1+s_2-1}b^{m_1+s_1-1}, \qquad A_2=\frac{b^{m_4-m_3+2}-a^{m_4-m_3+2}}{b^{-m_3}-a^{-m_3}},$$

$$A_3=b^{m_4-m_3+2}-\left\{\frac{b^{m_4-m_3+2}-a^{m_4-m_3+2}}{b^{-m_3}-a^{-m_3}}\right\}b^{-m_3}, \qquad A_4=\frac{T_a-T_b}{b^{-m_3}-a^{-m_3}},$$

$$A_5=T_b+\frac{(T_a-T_b)b^{-m_3}}{b^{-m_3}-a^{-m_3}}, \qquad Z_6=\frac{-[(m_2+1)(m_1+m_2)-m_1]}{(1-\sigma)(m_2+1)(m_1+m_2+1)+(\sigma m_1+\sigma-1)},$$

$$Z_1 = \frac{-(m_2 + m_4 - m_3 + 2)}{(1-\sigma)(m_2 + m_4 - m_3 + 3)(m_1 + m_2 + m_4 - m_3 + 3) + (\sigma m_1 + \sigma - 1)},$$

$$Z_2 = \frac{-(m_2 - m_3)}{(1-\sigma)(m_2 - m_3 + 1)(m_1 + m_2 - m_3 + 1) + (\sigma m_1 + \sigma - 1)},$$

$$Z_3 = \frac{-m_2}{(1-\sigma)(m_2 + 1)(m_1 + m_2 + 1) + (\sigma m_1 + \sigma - 1)},$$

$$Z_4 = \frac{-[(m_2 + m_4 - m_3 + 3)(m_1 + m_2 + m_4 - m_3 + 2) - m_1]}{(1-\sigma)(m_2 + m_4 - m_3 + 3)(m_1 + m_2 + m_4 - m_3 + 3) + (\sigma m_1 + \sigma - 1)},$$

$$Z_5 = \frac{-[(m_2 - m_3 + 1)(m_1 + m_2 - m_3) - m_1]}{(1-\sigma)(m_2 - m_3 + 1)(m_1 + m_2 - m_3 + 1) + (\sigma m_1 + \sigma - 1)}.$$

- For $m_3 = 0$

$$\begin{aligned}
\sigma_{rr} = \frac{E_0 \alpha_{t_0}}{A_1} \Bigg[\!\!\Bigg[& Q_2 \Big\{ Z_2 \Big[\Big(a^{m_1+s_2-1} b^{m_1+m_2+m_4-m_3+2} + \\
& - a^{m_1+m_2+m_4-m_3+2} b^{m_1+s_2-1} \Big) r^{m_1+s_1-1} + \Big(b^{m_1+s_1-1} a^{m_1+m_2+m_4-m_3+2} + \\
& - b^{m_1+m_2+m_4-m_3+2} a^{m_1+s_1-1} \Big) r^{m_1+s_2-1} + A_1 r^{m_1+m_2+m_4-m_3+2} \Big] + \\
& - \left(A_6 Z_8 + A_7 Z_3 \right) \Big[\Big(a^{m_1+s_2-1} b^{m_1+m_2} - a^{m_1+m_2} b^{m_1+s_2-1} \Big) r^{m_1+s_1-1} + \\
& + \Big(b^{m_1+s_1-1} a^{m_1+m_2} - b^{m_1+m_2} a^{m_1+s_1-1} \Big) r^{m_1+s_2-1} + A_1 r^{m_1+m_2} \Big] + \\
& - A_6 Z_3 \Big[\Big(a^{m_1+s_2-1} b^{m_1+m_2} \ln b - a^{m_1+m_2} b^{m_1+s_2-1} \ln a \Big) r^{m_1+s_1-1} + \\
& + \Big(b^{m_1+s_1-1} a^{m_1+m_2} \ln a - b^{m_1+m_2} a^{m_1+s_1-1} \ln b \Big) r^{m_1+s_2-1} + A_1 r^{m_1+m_2} \ln r \Big] \Big\} + \\
& + A_8 Z_8 \Big[\Big(a^{m_1+s_2-1} b^{m_1+m_2-m_3} - a^{m_1+m_2-m_3} b^{m_1+s_2-1} \Big) r^{m_1+s_1-1} + \\
& + \Big(b^{m_1+s_1-1} a^{m_1+m_2-m_3} - b^{m_1+m_2-m_3} a^{m_1+s_1-1} \Big) r^{m_1+s_2-1} + A_1 r^{m_1+m_2-m_3} \Big] + \\
& + A_8 Z_3 \Big[\Big(a^{m_1+s_2-1} b^{m_1+m_2} \ln b - a^{m_1+m_2} b^{m_1+s_2-1} \ln a \Big) r^{m_1+s_1-1} + \\
& + \Big(b^{m_1+s_1-1} a^{m_1+m_2} \ln a - b^{m_1+m_2} a^{m_1+s_1-1} \ln a \Big) r^{m_1+s_2-1} + A_1 r^{m_1+m_2} \ln r \Big] + \\
& + A_9 Z_3 \Big[\Big(a^{m_1+s_2-1} b^{m_1+m_2} - a^{m_1+m_2} b^{m_1+s_2-1} \Big) r^{m_1+s_1-1} + \\
& + \Big(b^{m_1+s_1-1} a^{m_1+m_2} - b^{m_1+m_2} a^{m_1+s_1-1} \Big) r^{m_1+s_2-1} + A_1 r^{m_1+m_2} \Big] \Bigg]\!\!\Bigg]
\end{aligned} \tag{5.31}$$

where

$$A_6 = \frac{b^{m_4+2} - a^{m_4+2}}{\ln\frac{b}{a}}, \qquad A_7 = b^{m_4+2} - \frac{\left(b^{m_4+2} - a^{m_4+2}\right)\ln b}{\ln\frac{b}{a}},$$

$$A_8 = \frac{T_b - T_a}{\ln\frac{b}{a}}, \qquad A_9 = T_b - \frac{\left(T_b - T_a\right)\ln b}{\ln\frac{b}{a}}.$$

Now

$$\sigma_{\theta\theta} = \frac{E_0\alpha_{t_0}}{A_1}\Bigg[\!\!\Bigg[Q_2\Bigg\{ Z_2\Bigg[\frac{(\sigma s_1 + 1 - \sigma)}{\left[(1-\sigma)s_1 + \sigma\right]}\Big(a^{m_1+s_2-1} b^{m_1+m_2+m_4-m_3+2} +$$
$$- a^{m_1+m_2+m_4-m_3+2} b^{m_1+s_2-1}\Big) r^{m_1+s_1-1} + \frac{(\sigma s_2 + 1 - \sigma)}{(1-\sigma)s_2 + \sigma]}\times$$
$$\times\Big(b^{m_1+s_1-1} a^{m_1+m_2+m_4-m_3+2} - b^{m_1+m_2+m_4-m_3+2} a^{m_1+s_1-1}\Big) r^{m_1+s_2-1}\Bigg] + Z_9 A_1 r^{m_1+m_2+m_4-m_3+2} +$$
$$-\left(A_6 Z_8 + A_7 Z_3\right)\Bigg[\frac{(\sigma s_1 + 1 - \sigma)}{\left[(1-\sigma))s_1 + \sigma\right]}\Big(a^{m_1+s_2-1} b^{m_1+m_2} - a^{m_1+m_2} b^{m_1+s_2-1}\Big) r^{m_1+s_1-1} +$$
$$+ \frac{(\sigma s_2 + 1 - \sigma)}{\left[(1-\sigma)s_2 + \sigma\right]}\Big(b^{m_1+s_1-1} a^{m_1+m_2} - b^{m_1+m_2} a^{m_1+s_1-1}\Big) r^{m_1+s_2-1}\Bigg] +$$
$$- A_1\left(Z_{10} A_6 + Z_{11} A_7\right) r^{m_1+m_2} - Z_{11} A_1 A_6 r^{m_1+m_2} \ln r +$$
$$- A_6 Z_3\Bigg[\frac{(\sigma s_1 + 1 - \sigma)}{\left[(1-\sigma)s_1 + \sigma\right]}\Big(a^{m_1+s_2-1} b^{m_1+m_2} \ln b - a^{m_1+m_2} b^{m_1+s_2-1} \ln a\Big) r^{m_1+s_1-1}$$
$$+ \frac{(\sigma s_2 + 1 - \sigma)}{\left[(1-\sigma)s_2 + \sigma\right]}\Big(b^{m_1+s_1-1} a^{m_1+m_2} \ln a - b^{m_1+m_2} a^{m_1+s_1-1} \ln b\Big) r^{m_1+s_2-1}\Bigg]\Bigg\} +$$
$$+\left(A_8 Z_8 + A_9 Z_3\right)\Bigg[\frac{(\sigma s_1 + 1 - \sigma)}{\left[(1-\sigma))s_1 + \sigma\right]}\Big(a^{m_1+s_2-1} b^{m_1+m_2} - a^{m_1+m_2} b^{m_1+s_2-1}\Big) r^{m_1+s_1-1} +$$
$$+ \frac{(\sigma s_2 + 1 - \sigma)}{\left[(1-\sigma)s_2 + \sigma\right]}\Big(b^{m_1+s_1-1} a^{m_1+m_2} - b^{m_1+m_2} a^{m_1+s_1-1}\Big) r^{m_1+s_2-1}\Bigg] + \qquad (5.32)$$
$$+ A_1\left(Z_{10} A_8 + Z_{11} A_9\right) r^{m_1+m_2} + Z_{11} A_1 A_8 r^{m_1+m_2} \ln r +$$
$$+ A_8 Z_3\Bigg[\frac{(\sigma s_1 + 1 - \sigma)}{\left[(1-\sigma)s_1 + \sigma\right]}\Big(a^{m_1+s_2-1} b^{m_1+m_2} \ln b - a^{m_1+m_2} b^{m_1+s_2-1} \ln a\Big) r^{m_1+s_1-1} +$$
$$+ \frac{(\sigma s_2 + 1 - \sigma)}{\left[(1-\sigma)s_2 + \sigma\right]}\Big(b^{m_1+s_1-1} a^{m_1+m_2} \ln a - b^{m_1+m_2} a^{m_1+s_1-1} \ln b\Big) r^{m_1+s_2-1}\Bigg]\!\!\Bigg]\!\!\Bigg]$$

where

$$Z_7 = \frac{-(m_2 + m_4 + 2)}{(1-\sigma)(m_2 + m_4 + 3)(m_1 + m_2 + m_4 + 3) + (\sigma m_1 + \sigma - 1)},$$

$$Z_8 = \frac{(1-\sigma)m_2^2 - m_1}{[(1-\sigma)(m_2 + 1)(m_1 + m_2 + 1) + (\sigma m_1 + \sigma - 1)]^2},$$

$$Z_9 = \frac{m_1 - (m_2 + m_4 + 3)(m_1 + m_2 + m_4 + 2)}{(1-\sigma)(m_2 + m_4 + 3)(m_1 + m_2 + m_4 + 3) + (\sigma m_1 + \sigma - 1)},$$

$$Z_{10} = \frac{-(m_1^2 + 2m_1m_2 + (1-\sigma)m_2^2 + m_1)}{[(1-\sigma)(m_2 + 1)(m_1 + m_2 + 1) + (\sigma m_1 + \sigma - 1)]^2},$$

$$Z_{11} = \frac{-(m_2^2 + m_1m_2 + m_2)}{(1-\sigma)(m_2 + 1)(m_1 + m_2 + 1) + (\sigma m_1 + \sigma - 1)}.$$

6. Particular case

In the final equations, if we take $m_1 = m_2 = m_3 = m_4 = 0$, we get the expressions for an isotropic and homogeneous hollow cylinder with uniform volumetric heat generation. This fact can be used as a validation of the problem.

Substituting $m_1 = m_2 = m_3 = m_4 = 0$ in Eq.(2.1), E, α_t, k and q become E_0, α_{t_0}, k_0 and q_0, which are the modulus of elasticity, thermal expansion coefficient, thermal conductivity and constant volumetric heat generation, respectively, for an isotropic and homogeneous material. From Eqs (5.3) and (5.12), we obtain $s_1 = 1$ and $s_2 = -1$.

The results obtained for thermal stresses with non-uniform heat generation are validated with the results as in Nayak and Mondal [6] by putting Q_1 and Q_2 equal to zero in the expressions (5.29) to (5.32), respectively.

7. Results and discussion

The mathematical thermoelastic model of a functionally graded thick hollow cylinder can be constructed by considering a thermal gradient through its radial direction.

A thick cylindrical vessel of the inner radius $a = 1m$ and outer radius $b = 1.2\ m$, Poisson's ratio $\sigma = 0.3$, thermal conductivity coefficient $k_0 = 2.09\ W / mK$ is considered to be constant. Material constants of the thermal expansion coefficient and modulus of elasticity are $\alpha_{t_0} = 10 \times 10^{-6} / {}^0C$ and $E_0 = 151GPa$ respectively. The inner surface of the hollow cylinder is fixed at $T_a = 10^0C$ and the outer surface is kept at $T_b = 0^0C$ and volumetric heat generation $q_0 = 500W / m^2$, which is constant.

8. Graphical illustrations

For $m_1 = m_2 = m_3 = m_4 \neq 0$, the temperature distribution obtained by (4.6) and thermal stress components obtained in Eqs (5.29) and (5.30) are used. For $m_1 = m_2 = m_3 = m_4 = 0$, the temperature distribution and thermal stress components are obtained by using the expressions (4.12), (5.31) and (5.32), respectively.

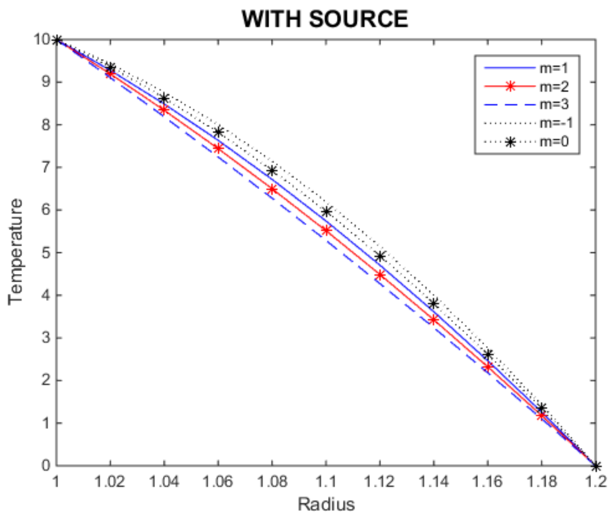
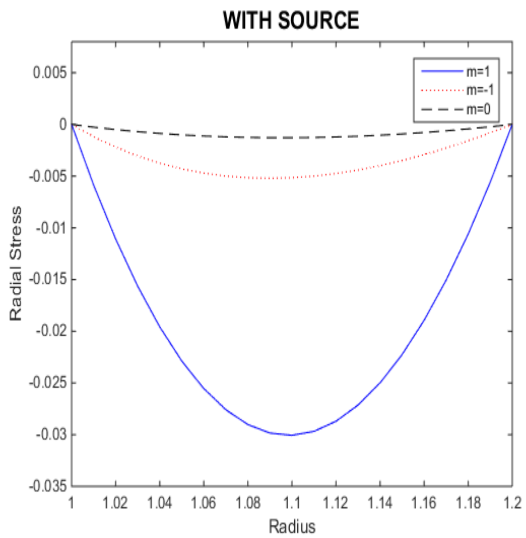


Fig.2. Variation of temperature distribution with the radius.



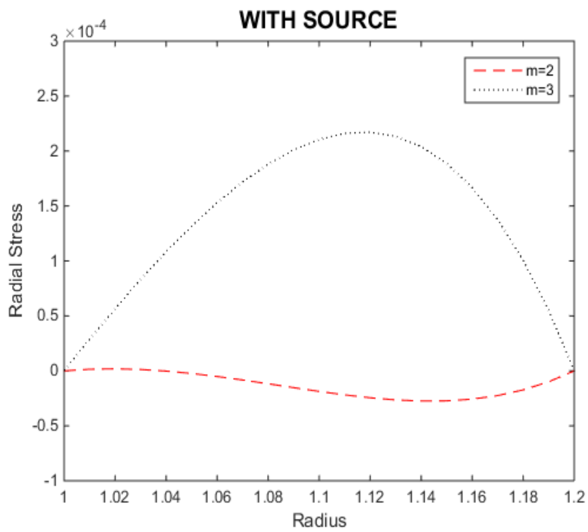


Fig.3. Variation of radial stress with the radius. Fig.4. Variation of radial stress with the radius.

The temperature distribution and thermal stresses with non-uniform heat generation are represented graphically and discussed as a particular case with variation in the power law index as $m = m_1 = m_2 = m_3 = m_4 = 0, 1, 2, 3, -1$.

Figure 2 represents the variation in the temperature with the radius in the presence of non-uniform heat source within the cylinder. For m =3,2,1,0 and -1, the temperature increases as the power law index decreases.

Figure 3 represents the radial stress distribution with heat generation within the cylinder. Due to assumed mechanical boundary conditions, the radial stress is zero on the surfaces and it is compressive throughout the cylinder. The compression shifts towards the outer surface for parameters $1, -1$ and 0. The variation can be observed in the case of the homogeneous and isotropic material for $m = 0$.

In Fig. 4, the radial stress becomes tensile in nature for $m = 3$ and the stress becomes tensile for the region up to the radius of about $r \approx 1.14$ and the remaining part is compressive for $m = 2$.

Figure 5 represents the circumferential stress distribution which increases as the power law index increases for $m = 0,1,2$ and 3. The stress increases from the inner to the outer surface. In Figure 6, it is interesting to note that for $m = -1$, the circumferential stress becomes tensile.

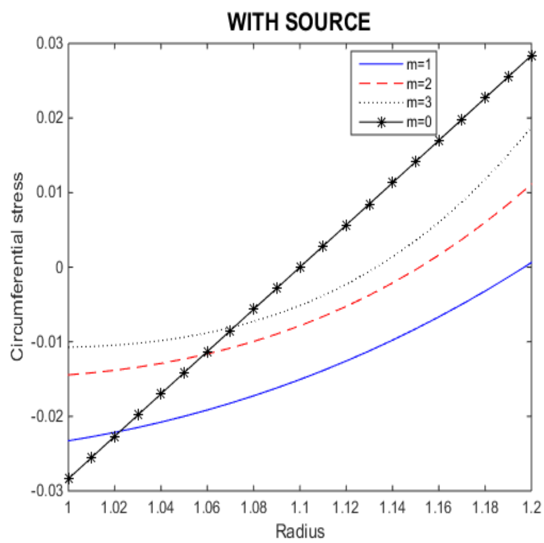


Fig.5. Variation of circumferential stress with the radius.

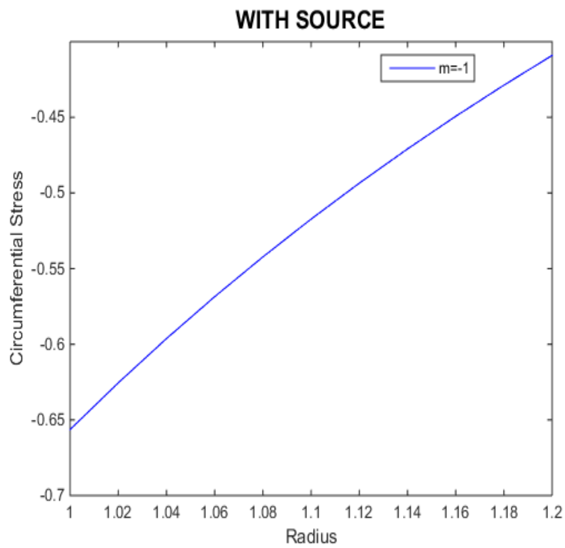


Fig.6. Variation of circumferential stress with the radius.

9. Conclusion

In this paper, exact analytical solutions are obtained for the temperature distribution and thermal stresses in an FGM hollow cylinder with a non-uniform internal heat generation. As a special case, a mathematical model is constructed for a hollow cylinder with material properties specified in the numerical calculations.

In this study, it is observed that the temperature, in the presence of the heat source, increases as the power law index decreases. The radial stress is compressive inside the cylinder as in the earlier results of Nayak and Mondal [6] but it is interesting to note that the radial stress becomes tensile for $m = 3$ and for $m = 2$, the region up to the radius of about $r \approx 1.14$ and the remaining part is compressive. For $m = 3,2,1$ and 0, the circumferential stress increases as the power law index increases from the inner to the outer surface and for $m = -1$, the circumferential stress becomes tensile.

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Nomenclature:

a – radius at the inner surface

b – radius at the outer surface

$\frac{d}{dr}$ – derivative with respect to the radial coordinate

E, E_0 – Young modulus
k, k_0 – thermal conductivity
q – non-uniform heat generation
t – time
T – temperature
T_a – temperature at the inner surface
T_b – temperature at the outer surface
u – radial coordinate
α_t, α_{t_0} – thermal expansion coefficient
ε_{rr} – radial strain
$\varepsilon_{\theta\theta}$ – circumferencial strain
σ – Poisson's ratio
σ_{rr} – radial stress
$\sigma_{\theta\theta}$ – circumferential stress

Subscripts

a – value at the inner surface
b – value at the outer surface

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